Dual Dig Level II (2006) - Solutions

1. Find the equation of the parabola in the usual *xy*-plane that passes through the points $(0,2)$, $(1,1)$, and $(-1,7)$.

   Solution: The generic form of the equation of a parabola is: $y = ax^2 + bx + c$.
   We solve the system:

$$\begin{cases} 2 = a(0)^2 + b(0) + c \\ 1 = a(1)^2 + b(1) + c \\ 7 = a(-1)^2 + b(-1) + c \end{cases} \qquad \begin{cases} 2 = c \\ 1 = a + b + c \\ 7 = a - b + c \end{cases}$$

   We know: $c = 2$. Focus on the last two equations now.

$$\begin{cases} 1 = a + b + 2 \\ 7 = a - b + 2 \end{cases} \qquad \begin{cases} a + b = -1 \\ a - b = 5 \end{cases}$$

   By using Substitution or the Addition / Elimination Method, we can solve the last system for *a* and *b*: $a = 2$, $b = -3$.

   The equation of the parabola is: $y = 2x^2 - 3x + 2$.

2. Find the equation (in factored form) of the cubic function of *x* that has 5, 2, and -4 as zeros and a graph with *y*-intercept 20.

   Solution: $y = a(x-5)(x-2)(x+4)$.

   Solve for *a*:

$$\begin{aligned} 20 &= a(0-5)(0-2)(0+4) \\ 20 &= a(-5)(-2)(4) \\ 20 &= 40a \\ a &= \frac{1}{2} \end{aligned}$$

   Equation: $y = \frac{1}{2}(x-5)(x-2)(x+4)$

3. Give the remainder when $2x^4+10x^2-6x+2$ is divided by $2x^2+6$.

Solution: The remainder is $-6x-10$; the remainder term is $\dfrac{-6x-10}{2x^2+6}$ or $-\dfrac{3x+5}{x^2+3}$.

$$
\begin{array}{rl}
 & \quad x^2 \qquad\qquad\quad + \ 2 \\
2x^2+0x+6 \,\big) & 2x^4+0x^3+10x^2-\ 6x+\ 2 \\
 & \cancel{2x^4+0x^3+\ 6x^2} \\
 & \underline{-2x^4-0x^3-\ 6x^2 \qquad\qquad\quad} \\
 & \qquad\qquad\quad 4x^2-\ 6x+\ 2 \\
 & \qquad\qquad\quad \cancel{4x^2+\ 0x+12} \\
 & \qquad\qquad\ \underline{-4x^2-\ 0x-12} \\
 & \qquad\qquad\qquad\quad -\ 6x-10
\end{array}
$$

4. It takes Larry 9 more hours than Ken to grade a stack of math Final Exams. Working together they can grade all of the papers in 20 hours. How long does it take Larry to grade the papers working alone?

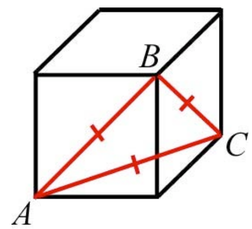
Solution: Let x = the length of time (in hours) it takes Ken to grade the papers working alone.
Let $x+9$ = the length of time (in hours) it takes Larry to grade the papers working alone.
Therefore, Ken grades $\dfrac{1}{x}$ of the stack per hour, and Larry grades $\dfrac{1}{x+9}$ of the stack per hour. The combined rate is $\dfrac{1}{x}+\dfrac{1}{x+9}$ of the stack per hour, which must equal $\dfrac{1}{20}$ of the stack per hour.

$$
\begin{aligned}
\frac{1}{x}+\frac{1}{x+9} &= \frac{1}{20} \\
20x(x+9)\left(\frac{1}{x}+\frac{1}{x+9}\right) &= 20x(x+9)\left(\frac{1}{20}\right) \\
20(x+9)+20x &= x(x+9) \\
20x+180+20x &= x^2+9x \\
40x+180 &= x^2+9x \\
0 &= x^2-31x-180 \\
x^2-31x-180 &= 0 \\
(x-36)(x+5) &= 0
\end{aligned}
$$

The only positive solution to the equation is: $x=36$.
However, our answer is: $x+9=45$ hours.

5. A, B, and C are vertices of a cube such that AB is a diagonal of one face of the cube, and BC is a diagonal of an adjacent face. Find the measure of Angle ABC in degrees.

Solution: Triangle ABC is an equilateral triangle, since side AC is also a diagonal of a face of the cube. Therefore, Angle ABC is 60°.



6. The fraction $\dfrac{2x^2+8x+6}{x^2+2x}$ can be expressed as a sum in the form $\dfrac{A}{x}+\dfrac{Bx+C}{x+2}$, where A, B, and C are integers. Find the value of C.

Solution: This resembles a "partial fraction decomposition" problem, although the given fraction is improper. Factor the denominator of the given fraction.

$$\frac{2x^2+8x+6}{x(x+2)}=\frac{A}{x}+\frac{Bx+C}{x+2}$$

$$x(x+2)\left[\frac{2x^2+8x+6}{x(x+2)}\right]=x(x+2)\left[\frac{A}{x}+\frac{Bx+C}{x+2}\right]$$

$$2x^2+8x+6=A(x+2)+x(Bx+C)$$

$$2x^2+8x+6=Ax+2A+Bx^2+Cx$$

$$2x^2+8x+6=(B)x^2+(A+C)x+(2A)$$

By matching the constant terms: $2A=6 \quad\Rightarrow\quad A=3$.
By matching the linear coefficients: $A+C=8 \quad\Rightarrow\quad 3+C=8 \quad\Rightarrow\quad C=5$.

7. The degree measures of the five interior angles of a pentagon are the first five terms in an arithmetic sequence. Find the measure of the third smallest angle (i.e., the "middle" angle) in degrees.

Solution: Let a be the measure of the third smallest angle in degrees.
Let d be the common difference of the arithmetic sequence.
The sum of the degree measures of the interior angles of the pentagon is:

$$(180^\circ)(5-2)=540^\circ.$$

We require:

$$(a-2d)+(a-d)+a+(a+d)+(a+2d)=540$$

$$5a=540$$

$$a=108^\circ \quad \text{(Answer)}$$

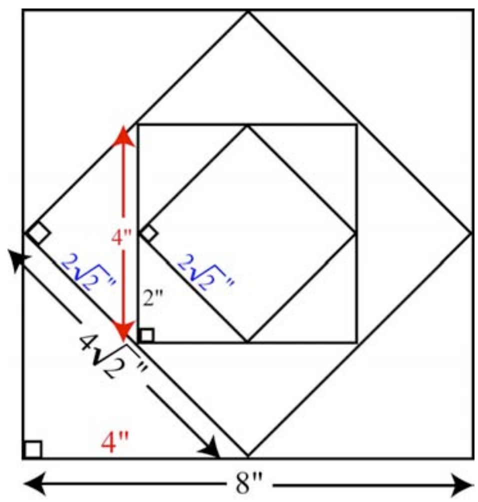
In fact, this is the common interior angle measure for a regular pentagon.

8. How many positive integers less than 100 have no positive odd integer divisors other than 1?

Solution: 7, namely the powers of 2 between 1 and 100, inclusive: 1, 2, 4, 8, 16, 32, and 64. (These correspond to $2^0,\ 2^1,\ 2^2,\ \ldots,\ 2^6$.) The prime factorizations of all other integers between 1 and 100 must have at least one odd positive prime factor. Note that 2 is the only even prime.

9. Consider a square with perimeter 32 inches. A 2nd (new) square is formed inside the original square by joining the midpoints of the original square. A 3rd square is formed by joining the midpoints of the 2nd square, and a 4th square is formed by joining the midpoints of the 3rd square. What is the total perimeter of all four squares?

Solution: The side lengths of the squares are 8, $4\sqrt{2}$, 4, and $2\sqrt{2}$ inches.
The total of the perimeters is: $4\left(8+4\sqrt{2}+4+2\sqrt{2}\right)=4\left(12+6\sqrt{2}\right)=48+24\sqrt{2}$ inches.



10. Find the determinant: $\begin{vmatrix} 3 & 12 & -90 \\ 0 & 4 & 700 \\ 0 & 0 & 5 \end{vmatrix}$.

Solution: By expanding by cofactors along the first column, we find that the determinant is simply the product of the entries along the main diagonal: $(3)(4)(5)=60$.

11. Factor completely as a product of linear factors: x^3-3+3x^2-x.

Solution:

$$\begin{aligned} x^3-3+3x^2-x &= x^3+3x^2-x-3 && \text{(Rewriting in descending powers of } x.) \\ &= \left(x^3+3x^2\right)+\left(-x-3\right) && \text{(We will factor by grouping.)} \\ &= x^2\left(x+3\right)-\left(x+3\right) \\ &= \left(x^2-1\right)\left(x+3\right) \\ &= \left(x+1\right)\left(x-1\right)\left(x+3\right) \end{aligned}$$

12. Tweedledee randomly tosses two fair coins and then (truthfully) tells Tweedledum that at least one of them came up “heads.” Given this information, what is the probability that both coins came up “heads”?

Solution: Consider ordered pairs of the form (first coin result, second coin result).
There are four equally likely possibilities: (H, H), (H, T), (T, H), and (T, T).
It is known that at least one of the coins has come up “heads,” so the (T, T) pair must be eliminated from consideration. The other three possibilities are still equally likely, however. Since only one of them corresponds to the situation where both coins come up “heads,” the answer is: $\frac{1}{3}$.

13. The five students in a class take a test, and their average score is 64 points. After discovering that the top-scoring student had cheated, the teacher computes the average of the four other scores and finds that that average is 60 points. What score did the cheating student originally get on the test?

Solution: The five students originally had a combined score of $64 \cdot 5 = 320$ points.
The four (presumably) non-cheating students had a combined score of $60 \cdot 4 = 240$.
Therefore, the cheating student originally had a score of $320 - 240 = 80$ points.

14. A diagonal of a regular polygon is any line segment whose endpoints are two distinct vertices of the polygon and is not a side of the polygon. How many diagonals does a 16-sided regular polygon have?

Solution: Each of the 16 vertices is an endpoint for 13 diagonals, since a diagonal cannot connect a vertex with itself or with its two adjacent neighbors. Therefore, there are $16 \times 13 = 208$ vertex-diagonal incidences. However, the diagonal from a vertex *A* to another vertex *B* is also the diagonal from *B* to *A*, so we must divide by 2 to avoid double-counting the diagonals. The answer is given by: $\frac{208}{2} = 104$ diagonals or

$$\frac{(16)(13)}{2} = 8(13) = 104 \text{ diagonals}.$$

15. f is a linear function such that $f(x) = mx + b$ for some real numbers *m* and *b*. It is known that f is its own inverse, but it is **not** the identity function given by $f(x) = x$. What must *m* be?

Solution: $m = -1$.

Method 1 (Graphical approach)

The graph of f must be symmetric about the line $y = x$, which is the line of reflection for graphs of inverse functions. Therefore, because $y = x$, itself, is excluded, the graph of f must be a line that is perpendicular to the line $y = x$, which has slope 1. The slope of such a line must be -1, the negative reciprocal of 1. Therefore, *m*, the slope of the graph of f, must be -1.

Method 2 ("Analytical" approach)

We require: $f\left(f\left(x\right)\right)=x$ for all real x.

$f\left(f\left(x\right)\right)=f\left(mx+b\right)=m\left(mx+b\right)+b=m^2x+mb+b=\left(x\right)m^2+\left(b\right)m+\left(b\right)$

$f\left(f\left(x\right)\right)=x$ implies:

$$\left(x\right)m^2+\left(b\right)m+\left(b\right)=x$$
$$\left(x\right)m^2+\left(b\right)m+\left(b-x\right)=0$$

Using the Quadratic Formula, we get:

$$\begin{aligned} m &= \frac{-b\pm\sqrt{b^2-4\left(x\right)\left(b-x\right)}}{2x} \\ &= \frac{-b\pm\sqrt{b^2-4xb+4x^2}}{2x} \\ &= \frac{-b\pm\sqrt{\left(b-2x\right)^2}}{2x} \\ &= \frac{-b\pm\left|b-2x\right|}{2x} \\ &= \frac{-b\pm\left(b-2x\right)}{2x} \end{aligned}$$

The "+" case yields $m=-1$.

The "$-$" case yields $m=\frac{x-b}{x}=1-\frac{b}{x}$, which is constant for nonzero x if and only if $b=0$ and, therefore, $m=1$. However, that would correspond to the identity function given by $f\left(x\right)=x$, which we have explicitly excluded.

16. Simplify the number: $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$. Hint: Consider the square of the number.

Solution: Let x represent the number $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}$.

Then, $x^2=1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$, which equals $1+x$.

Solve for x:

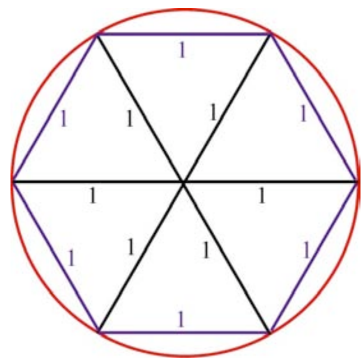
$$x^2=1+x$$
$$x^2-x-1=0$$

By the Quadratic Formula, the only positive solution for x is: $x=\frac{1+\sqrt{5}}{2}$.

Note: This very important number, seen in art and nature, is called the **golden mean** or **golden ratio**, denoted by the Greek letter ϕ (phi).

17. Find the difference between the circumference of a circle of radius 1 centimeter and the perimeter of a regular hexagon inscribed within the circle. (By “inscribed,” we mean that all six vertices of the hexagon lie on the circle.) Round off your answer to the nearest hundredth of a centimeter.

Solution: The circumference of the circle is given by $2\pi \approx 6.28$ cm.
The perimeter of the hexagon is 6 cm; the six small triangles drawn below are all equilateral:



The approximate difference is given by: $6.28 - 6 = 0.28$ cm.

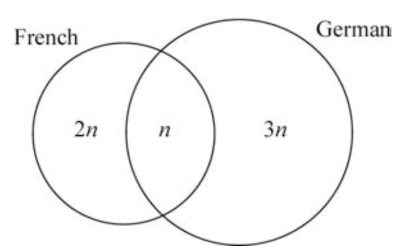
18. Find the digit in the ones place of the number given by: $0! + 1! + 2! + ... + 2006!$

Solution:

Observe that $5!$ and all further factorials are divisible by 10, since 2 and 5 are both factors of all of those factorials. Therefore, the answer is the ones digit for the number $0! + 1! + 2! + 3! + 4! = 1 + 1 + 2 + 6 + 24 = 34$. The answer is 4.

19. At a particular school, each student must study French and/or German. $\frac{1}{3}$ of the students who study French also study German, and $\frac{1}{4}$ of the students who study German also study French. What fraction of the students at the school study German?

Solution: Let n = the number of students at the school taking both French and German.
Then, $3n$ students study French, and $4n$ students study German.
Draw a Venn diagram:



The fraction of students who study German is: $\frac{4n}{6n} = \frac{2}{3}$.

20. Three players, A, B, and C, take turns rolling a standard six-sided die. The game ends as soon as somebody rolls a "6" on the die, in which case that player wins the game. Player A rolls first, then B, then C, then A rolls again, and so forth, until someone wins. What is the exact probability that Player A will win this game?

Solution: Player A can win on rolls 1, 4, 7, 10, etc.

The probability that A wins on roll 1 is $\frac{1}{6}$.

The probability that A wins on roll 4 is $\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)$, because that is the probability of the three players failing to get a "6" on the first three rolls and A getting a "6" on the fourth roll.

The probability that A wins on roll 7 is $\left(\frac{5}{6}\right)^6\left(\frac{1}{6}\right)$, because that is the probability of the three players failing to get a "6" on the first six rolls and A getting a "6" on the seventh roll.

The probability that A wins is:

$$\frac{1}{6}+\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^6\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^9\left(\frac{1}{6}\right)+\ldots = \frac{1}{6}\left[1+\left(\frac{5}{6}\right)^3+\left(\frac{5}{6}\right)^6+\left(\frac{5}{6}\right)^9+\ldots\right]$$

Use the formula for the sum of a geometric series:

$$= \frac{1}{6}\left[\frac{1}{1-\left(\frac{5}{6}\right)^3}\right]$$

$$= \frac{1}{6}\left[\frac{1}{1-\frac{125}{216}}\right]$$

$$= \frac{1}{6}\left[\frac{1}{\frac{91}{216}}\right]$$

$$= \frac{1}{6}\left[\frac{216}{91}\right]$$

$$= \frac{36}{91}$$