Dual Dig Level I (2013) – Solutions

1. A wooden cube has a volume of 27 in^3 and is painted black on all sides. A very, very thin saw blade is used to cut the original cube into 27 cubes of side length 1 inch. What fraction of the cubes will have NO black paint on them whatsoever?

   Answer: 1/27 (due to the cube in the exact center of the original cube)

2. Simplify as a fraction: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)$

   Answer: 3/5

   Solution: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)=$
   $\left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{15}{16}\right)\left(\frac{24}{25}\right)=\left(\frac{3}{1}\right)\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)\left(\frac{3}{25}\right)=\left(\frac{1}{1}\right)\left(\frac{1}{1}\right)\left(\frac{1}{1}\right)\left(\frac{3}{5}\right)=\frac{3}{5}$

3. On Sunday, a gas company promises to increase its gas prices by 200% by that Friday at noon. If the company raises prices by 50% by Wednesday at noon, by what percent must prices rise from Wednesday at noon to Friday at noon if the company is to keep its promise?

   Answer: 100%.

   Solution 1: The easiest way to see this is to assume that gas prices start at \$1/gallon (LOL). A 50% increase takes the price to \$1.50/gallon on Wednesday. Then, a 100% increase (or a doubling) will have to take place by Friday in order to get the price to \$3/gallon, which represents a 200% increase over \$1/gallon.

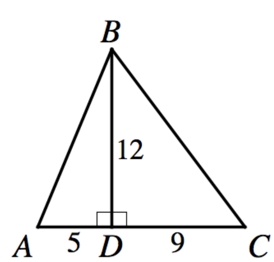
   Solution 2: Assume that the Sunday price is x. The company promises to raise the price to $3x$ by Friday. Solve: $(1 + a)(1.5x) = 3x$ for a. We obtain $a = 1$, corresponding to a 100% increase.

4. Factor using integer coefficients: $2x^3-5x^2+6x-15$

   Answer: $(x^2+3)(2x-5)$

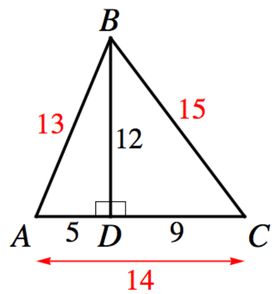
   Solution: Factor by grouping. $2x^3-5x^2+6x-15=x^2(2x-5)+3(2x-5)=(x^2+3)(2x-5)$

5. What is the perimeter of triangle ABC below? $\overline{AD}$ is 5 inches long, $\overline{BD}$ is 12 inches long, and $\overline{CD}$ is 9 inches long. $\overline{BD}$ is perpendicular to $\overline{AC}$.



   Answer: 42 inches.

Solution: The Pythagorean Theorem can be used to find the hypotenuses of triangles *ADB* and *CDB*. Fortunately, we have well-known Pythagorean triples for the side lengths: 5-12-13 and 9-12-15 (this triangle is similar to the 3-4-5 triangle).



6. Solve for R: $2aR - 3R = 5bR + 6b$

   Answer: $\dfrac{6b}{2a-3-5b}$

   Solution: $2aR - 3R - 5bR = 6b$ ➔ $R(2a - 3 - 5b) = 6b$ ➔ $R = \dfrac{6b}{2a-3-5b}$

7. Give the remainder when $3x^3 - 2x + 4$ is divided by $x^2 - 1$.

   Answer: Remainder = $x + 4$.

   Solution: Using <u>long division</u>, the quotient is $3x$. Placeholder zeros in both the dividend and divisor may help.

8. A Norman window (depicted below) consists of a semicircular region with diameter forming the top of a rectangle of width x. If the height of the rectangular part of the window is 6 feet less than four times the width, write an expression in x for the <u>perimeter</u> of the entire window (in feet).

   Answer: $P = \dfrac{\pi}{2}x + 9x - 12$ (in feet),

   Solution: Let x = the width of the rectangle. Then, its height = $4x - 6$, and the radius of the semicircle is: $\dfrac{x}{2}$.

   The perimeter of the semicircle = $\dfrac{1}{2} \bullet 2\pi r = \pi \bullet \dfrac{x}{2} = \dfrac{\pi x}{2}$ (in feet), while the <u>partial</u> perimeter of the rectangle = $x + 2(4x - 6) = 9x - 12$ (in feet). (You can't use the width x twice, because one side of the rectangle is in the 'interior' of the shape.) Add these two expressions together and you get the answer.

9. Three generations of the Foster family, two members from each generation, are going to the Del Mar Fair. The two members of the youngest generation receive a 50% discount as children. The two members of the oldest generation receive a 25% discount as senior citizens. The two members of the middle generation receive no discount. Grandfather Foster, whose senior ticket costs $6.00, is paying for everyone. How many dollars must he pay?

Answer: $36

Solution: Grandfather Foster's ticket costs $6, which is $\frac{3}{4}$ of the full price, so each ticket (at full price) costs $\frac{4}{3} \bullet 6 = \$8$. Each child's ticket costs $\frac{1}{2} \bullet 8 = \$4$. The cost of all the tickets is: $2(6+8+4) = \$36$.

10. Simplify by writing as a single non-compound (non-complex) fraction: $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+x}}}$

Answer: $\frac{3x+5}{2x+3}$

Solution: $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+x}}} \rightarrow 1+\cfrac{1}{1+\cfrac{1}{\cfrac{x+2}{1+x}}} \rightarrow 1+\cfrac{1}{1+\cfrac{x+1}{x+2}} \rightarrow 1+\cfrac{1}{\cfrac{2x+3}{x+2}} \rightarrow 1+\frac{x+2}{2x+3}$

$\rightarrow \frac{3x+5}{2x+3}$

11. How many liters of pure water must be added to 12 liters of 30% antifreeze solution to produce a 20% solution?

Answer: 6 liters of water

NOTE: Pure water contains 0% antifreeze.

Solution: Let x = liters of water needed. $0(x) + .30(12) = .20(x + 12)$, then solve.

12. For $f(x) = \frac{2}{x+1}$, find the difference quotient: $\frac{f(x+h)-f(x)}{h}$ in simplest form.

Answer: $-\frac{2}{(x+h+1)(x+1)}$

Solution: $\cfrac{\cfrac{2}{x+h+1}-\cfrac{2}{x+1}}{h} \rightarrow \cfrac{\cfrac{2(x+1)-2(x+h+1)}{(x+h+1)(x+1)}}{h} \rightarrow \frac{2x+2-2x-2h-2}{h(x+h+1)(x+1)}$

$\rightarrow \frac{-2h}{h(x+h+1)(x+1)} \rightarrow -\frac{2}{(x+h+1)(x+1)}$.

13. Find the equation of a parabola (in the form $y = ax^2 + bx + c$) with vertex at $(3, -8)$ and passing through the point $(6, 10)$.

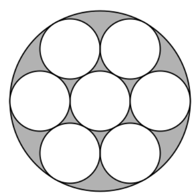
Answer: $y = 2x^2 - 12x + 10$

Solution 1: Since $(3, -8)$ is the vertex, we can substitute this into vertex form: $y = a(x - 3)^2 - 8$. Furthermore, we can substitute in the point $(6, 10) \rightarrow 10 = a(6 - 3)^2 - 8$. Solve and we get: $a = 2$. Now the equation is: $y = 2(x - 3)^2 - 8$. All that is left is to multiply out and get the stated answer.

Solution 2: Since $(3, -8)$ is the vertex, $x = 3$ is the axis of symmetry, meaning that $(0, 10)$ is the reflection of $(6, 10)$; it is also the y-intercept of the parabola, and $c = 10$. Substitute in the points $(6, 10)$ and $(3, -8)$ into $y = ax^2 + bx + 10$. Solving the system $\begin{cases} 10 = a(6)^2 + b(6) + 10 \\ -8 = a(3)^2 + b(3) + 10 \end{cases}$,

or $\begin{cases} 10 = 36a + 6b + 10 \\ -8 = 9a + 3b + 10 \end{cases}$, yields $a = 2$ and $b = -12$ and thus the answer.

14. Find the sum of the areas of the shaded regions below. The seven white circular disks each has radius 1 inch. The innermost white disk is tangent to the six surrounding white disks, and those six disks are each tangent to the larger circle and to three neighboring white disks.



Answer: 2π square inches

Solution: The radius of the large circle is 3 inches, since a radius can be constructed from a radius of the innermost circle and a diameter (or two radii) of one of its neighbors. The total area of the large circle is: $\pi\left(r_{\text{large}}\right)^2 = \pi(3)^2 = 9\pi$ square inches. Each of the small white circular disks has area $\pi\left(r_{\text{small}}\right)^2 = \pi(1)^2 = \pi$ square inches, for a total of 7π square inches for all seven white disks. The remaining (shaded) area is: $9\pi - 7\pi = 2\pi$ square inches.

15. A sequence of a_n terms, starting with a_1, is defined by: $a_n = (-1)^{n+1} \cdot n$. The first six terms, for example, are given by: $1, -2, 3, -4, 5, -6$. Find the sum of the first 1000 terms of the sequence.

Answer: -500

Solution: Observe that 1 and -2 add up to -1, 3 and -4 add up to -1, etc. We can partition the first 1000 terms into 500 pairs of terms, each with a sum of -1. The grand total is given by: $(-1)(500) = -500$.

16. Ken competes in a triathlon. He averages 2 miles per hour in the 1/4-mile swim and 6 miles per hour in the 3-mile run. His goal is to finish the triathlon in 2 hours. To accomplish his goal, what must his average speed (in mph) be for the 15-mile bicycle ride?

    Answer: 120/11 mph, or $10\frac{10}{11}$ mph

    Solution: Ken can complete the swim in $\frac{\frac{1}{4}}{2}=\frac{1}{8}$ of an hour. He can complete the run in $\frac{3}{6}=\frac{1}{2}$ of an hour. Subtracting these from 2 hours leaves $\frac{11}{8}$ hours to complete the bicycle ride. His average speed for the ride must be $\frac{15}{\frac{11}{8}}=\frac{120}{11}$ miles per hour.

17. In the magic square shown, the sums of the numbers in each row, column, and diagonal are the same. Five of the numbers are represented by *v*, *w*, *x*, *y*, and *z*. Find $y+z$.

v	24	w
18	x	y
25	z	21

    Answer: 46.

    Solution: Since *v* appears in 1 row, 1 column, and 1 diagonal, the sum of the remaining numbers in each must be the same. Thus, $25 + 18 = 24 + w = 21 + x$ ➔ $w = 19$ and $x = 22$. This means that the sum of all entries in the diagonal is 66. It follows that $v = 23$, $y = 26$, and $z = 20$. $y + z = 46$.

18. Give one solution (x, y, z) to the system: $\begin{cases} x^3+y^3+z^3=495 \\ x+y+z=15 \\ xyz=105 \end{cases}$, where *x*, *y*, and *z* are positive integers.

    Answer: The integers are 3, 5, and 7 in no particular order. $(3,5,7)$ is a solution.

    Solution: A prime factorization of 105 shows that: $105 = 3\bullet 5\bullet 7$, thus showing that $(3,5,7)$ (and other ordered triples representing permutations of those numbers) will solve the third equation. These triples also satisfy the other equations. The prime factorization is unique (up to order), so there are no other solutions.

19. You stand before three boxes, each with two coins. One box contains one gold coin and one silver coin. Another contains two gold coins. Another contains two silver coins. You do not know which box is which. You randomly select a box and randomly take out one of the coins. You see that the coin is gold. What is the probability that the other coin in the selected box is also gold?

    Answer: 2/3.

    Comment: (This is Bertrand's Box Paradox. See *Wikipedia*.)

    Explanation 1: The probability that a box has two coins of the same type is 2/3.

Explanation 2: There are three equally likely ways of picking a gold coin, say G1 and G2 from the box with two gold coins ("Box GG") and G3 from the box with one gold coin and one silver coin ("Box GS"). We are twice as likely to pick a gold coin from Box GG as from Box GS, it is impossible to pick a gold coin from Box SS, and the probabilities for the three boxes must add up to 1. Therefore, the probability that the gold coin is from Box GG is 2/3. This paradox is similar to the Monty Hall (or "Three Door") Problem in that people have a bad habit of automatically assigning equal probabilities to remaining possibilities.

20. Find the greatest integer exponent *n* for which 2013! is divisible by 2^n (meaning $2^n \cdot k = 2013!$ for some positive integer *k*). 2013! is "2013 factorial," which is obtained from: $(1)(2)(3)(4)\cdots(2013)$.

Answer: 2004

Solution: 2013! is the product of the integers from 1 through 2013. If *n* is a positive integer, then $\left\lfloor \frac{2013}{2^n} \right\rfloor$ is the number of those integers divisible by 2^n, where $\lfloor\ \rfloor$ is the "floor" or "round down" operator. (1024 is the highest power of 2 that is no higher than 2013.)
For example, 1006 of those integers are divisible by 2, 503 of them are divisible by 4, etc.
The answer is given by:

$$\left\lfloor \frac{2013}{2} \right\rfloor + \left\lfloor \frac{2013}{4} \right\rfloor + \left\lfloor \frac{2013}{8} \right\rfloor + \left\lfloor \frac{2013}{16} \right\rfloor + \left\lfloor \frac{2013}{32} \right\rfloor$$

$$+ \left\lfloor \frac{2013}{64} \right\rfloor + \left\lfloor \frac{2013}{128} \right\rfloor + \left\lfloor \frac{2013}{256} \right\rfloor + \left\lfloor \frac{2013}{512} \right\rfloor + \left\lfloor \frac{2013}{1024} \right\rfloor$$

$$= 1006 + 503 + 251 + 125 + 62$$

$$+ 31 + 15 + 7 + 3 + 1$$

$$= 2004$$