Example Set 5 (Factoring Polynomials)

Factor the following polynomials over the integers.

a) $x^2 + 9x + 20$

b) $x^2 - 20x + 100$ (Hint: This is a Perfect Square Trinomial (PST).)

c) $x^2 - 4x - 12$

d) $3x^2 - 20x - 7$

e) $4x^2 + 11x + 6$

f) $2x^2 + 10x + 5$

g) $-3x^2 + 6x - 3$

h) $x^4 - 16$

i) $a^3 - 3a + 2a^2b - 6b$
(Hint: Use Factoring by Grouping. This is when we group terms and factor each group "locally" before we factor the entire expression "globally" by factoring out the GCF.)

j) $4x^2 + 9y^2$

k) $x^3 + 125y^3$

l) $x^3 - 125y^3$

§ Solution

a) $ax^2 + bx + c = x^2 + 9x + 20 = (x+5)(x+4)$

We want 5 and 4, because they have product = c = 20 and (since this is the $a = 1$ case) sum = b = 9. We can rearrange the factors: $(x+4)(x+5)$.

b) $\underbrace{x^2}_{(x)^2} - 20x \underbrace{+100}_{(10)^2} = \underbrace{(x-10)^2}_{\text{Check: } 2(x)(-10) = -20x}$, or $(x-10)(x-10) = (x-10)^2$

Guess that this is a PST for now.

c) $x^2 - 4x - 12 = (x-6)(x+2)$

How do we know we need -6 and $+2$?

The constant term, c, is negative, so use opposite signs: one "+" and one "$-$."

The middle coefficient, b, is negative, so the negative number must be higher in absolute value than the positive number; it "carries more weight."

2-factorizations of –12 (which is c) Think: What? • What?? = –12.	Sum = b = -4? ($a = 1$ case)
$-$ 12, +1	No
$-$ 6, +2	**Yes – Can stop**
$-$ 4, +3	No

d) $F + (O + I) + L = 3x^2 - 20x - 7 = (3x+1)(x-7)$

F = First product (product of the First terms)
O = Outer product (product of the Outer terms)
I = Inner product (product of the Inner terms)
L = Last product (product of the Last terms)

$(3x \quad\quad)(x \quad\quad) \leftarrow F = 3x^2$; factors must be $3x$ and x

$\downarrow$ Need $L = -7$

$+7 \quad\quad -1$

$-1 \quad\quad +7 \quad \leftarrow$ Makes $O + I = 20x$. We need $O + I$ to be $-20x$, which is the middle term of the trinomial. We're only off by a sign, so we change both signs.

$+\mathbf{1} \quad\quad -\mathbf{7} \quad \leftarrow$ Makes $O + I = -20x$. This works.

$-7 \quad\quad +1$

Also, $b = -20$, a "very negative" coefficient, so we are inclined to pair up the $3x$ and the -7 to form the outer product, since they form $-21x$.

e) $4x^2 + 11x + 6 = (4x+3)(x+2)$

Method 1: Trial-and-Error ("Guess") Method

$$\begin{array}{ll} (\quad\quad &)(\quad\quad) \\ \mathbf{4x} & \mathbf{x} \\ 2x & 2x \end{array} \Bigg\} F = 4x^2$$

$$\left.\begin{array}{ll} +1 & +6 \\ +6 & +1 \\ +2 & +3 \\ +\mathbf{3} & +\mathbf{2} \end{array}\right\} L = 6\text{; need both "+" because of } +11x$$

Method 2: Factoring by Grouping

4 and 6 are neither prime nor "1," so we may prefer this method.
We want two integers whose product is ac = (4)(6) = 24
and whose sum is b = 11. We want 8 and 3; split the middle
term accordingly.

$$4x^2 + 11x + 6 = 4x^2 \underbrace{+\ 8x + 3x}_{\text{OK to switch}} + 6$$

$$= (4x^2 + 8x) + (3x + 6) \quad \leftarrow \text{Group terms}$$

$$= 4x(x+2) + 3(x+2) \quad \leftarrow \text{"Local factoring"}$$

$$= (4x+3)(x+2) \quad \leftarrow \text{"Global factoring"}$$

f) $2x^2 + 10x + 5$ is **prime** or **irreducible over the integers** (i.e., it cannot be broken down further using integer coefficients). None of these combinations work:

$$(2x \quad\quad)(x \quad\quad) \leftarrow F = 2x^2\text{; factors must be } 2x \text{ and } x$$

$$\downarrow \text{Need } L = 5\text{; need both "+" because of } +10x$$

$$\begin{array}{ll} +1 & +5 \\ +5 & +1 \end{array}$$

We could also apply the **Test for Factorability**. The **discriminant** $b^2 - 4ac = (10)^2 - 4(2)(5) = 100 - 40 = 60$, which is **not** a perfect square, and the GCF = 1, so the polynomial is **prime**.

g)

$$-3x^2 + 6x - 3 = \underbrace{-3}_{\text{GCF}} \underbrace{(x^2 - 2x + 1)}_{\text{a PST}}$$

You should usually factor out the GCF first.

$$= -3(x-1)^2$$

h) Apply the Difference of Two Squares formula $\left[a^2 - b^2 = (a+b)(a-b)\right]$ twice:

$$\underbrace{x^4}_{(x^2)^2} - \underbrace{16}_{(4)^2} = \underbrace{\left(x^2+4\right)}_{\text{prime}}\left(\underbrace{x^2}_{(x)^2} - \underbrace{4}_{(2)^2}\right)$$
$$= \left(x^2+4\right)(x+2)(x-2)$$

i) Use Factoring by Grouping:

$$a^3 - 3a + 2a^2b - 6b = \left(a^3 - 3a\right) + \left(2a^2b - 6b\right)$$
$$= a\left(a^2-3\right) + 2b\left(a^2-3\right)$$
$$= (a+2b)\left(a^2-3\right)$$

j) $4x^2 + 9y^2$ is **prime**. The GCF = 1, and we have no formula for the Sum of Two Squares (for now…; this will change when we discuss imaginary numbers in Section 2.1).

k) Apply the Sum of Two Cubes formula $\left[a^3 + b^3 = \underbrace{\underbrace{(a+b)}_{\text{"Expected factor"}}\left(a^2 \underbrace{-ab}_{\text{NOT } -2ab} + b^2\right)}_{\text{The visible signs follow the pattern: same, different, "+"}}\right]$:

$$\underbrace{x^3}_{(x)^3} + \underbrace{125y^3}_{(5y)^3} = (x+5y)\left(x^2 - 5xy + 25y^2\right)$$

l) Apply the Difference of Two Cubes formula $\left[a^3 - b^3 = \underbrace{\underbrace{(a-b)}_{\text{"Expected factor"}}\left(a^2 \underbrace{+ab}_{\text{NOT } +2ab} + b^2\right)}_{\text{The visible signs follow the pattern: same, different, "+"}}\right]$:

$$\underbrace{x^3}_{(x)^3} - \underbrace{125y^3}_{(5y)^3} = (x-5y)\left(x^2 + 5xy + 25y^2\right).$$ §