

Example Set 5 (Factoring Polynomials)

Factor the following polynomials over the integers.

a) $x^2 + 9x + 20$

b) $x^2 - 20x + 100$ (Hint: This is a Perfect Square Trinomial (PST).)

c) $x^2 - 4x - 12$

d) $3x^2 - 20x - 7$

e) $4x^2 + 11x + 6$

f) $2x^2 + 10x + 5$

g) $-3x^2 + 6x - 3$

h) $x^4 - 16$

i) $a^3 - 3a + 2a^2b - 6b$

(Hint: Use Factoring by Grouping. This is when we group terms and factor each group “locally” before we factor the entire expression “globally” by factoring out the GCF.)

j) $4x^2 + 9y^2$

k) $x^3 + 125y^3$

l) $x^3 - 125y^3$

§ Solution

a) $ax^2 + bx + c = x^2 + 9x + 20 = (x + 5)(x + 4)$

We want 5 and 4, because they have product = $c = 20$ and (since this is the $a = 1$ case) sum = $b = 9$. We can rearrange the factors: $(x + 4)(x + 5)$.

b) $\underbrace{x^2}_{(x)^2} - \underbrace{20x}_{(10)^2} + \underbrace{100}_{(10)^2} = \underbrace{(x - 10)^2}_{\text{Check: } 2(x)(-10) = -20x}$, or $(x - 10)(x - 10) = (x - 10)^2$
 Guess that this is a PST for now.

c) $x^2 - 4x - 12 = (x - 6)(x + 2)$

How do we know we need -6 and $+2$?

The constant term, c , is negative, so use opposite signs: one "+" and one "-."

The middle coefficient, b , is negative, so the negative number must be higher in absolute value than the positive number; it "carries more weight."

2-factorizations of -12 (which is c) Think: What? • What?? = -12 .	Sum = $b = -4$? ($a = 1$ case)
$-12, +1$	No
$-6, +2$	Yes – Can stop
$-4, +3$	No

d) $F + (O + I) + L = 3x^2 - 20x - 7 = (3x + 1)(x - 7)$

F = First product (product of the First terms)

O = Outer product (product of the Outer terms)

I = Inner product (product of the Inner terms)

L = Last product (product of the Last terms)

$(3x \quad)(x \quad) \leftarrow F = 3x^2$; factors must be $3x$ and x

\downarrow Need $L = -7$

$+7 \quad -1$

$-1 \quad +7 \leftarrow$ Makes $O + I = 20x$. We need $O + I$ to be $-20x$,

which is the middle term of the trinomial.

We're only off by a sign, so we change both signs.

$+1 \quad -7 \leftarrow$ Makes $O + I = -20x$. This works.

$-7 \quad +1$

Also, $b = -20$, a "very negative" coefficient, so we are inclined to pair up the $3x$ and the -7 to form the outer product, since they form $-21x$.

$$e) \quad 4x^2 + 11x + 6 = (4x + 3)(x + 2)$$

Method 1: Trial-and-Error ("Guess") Method

$$\begin{array}{cc} (&) \\ \mathbf{4x} & \mathbf{x} \\ 2x & 2x \end{array} \left. \vphantom{\begin{array}{cc} (&) \\ \mathbf{4x} & \mathbf{x} \\ 2x & 2x \end{array}} \right\} F = 4x^2$$

$$\begin{array}{cc} +1 & +6 \\ +6 & +1 \\ +2 & +3 \\ +\mathbf{3} & +\mathbf{2} \end{array} \left. \vphantom{\begin{array}{cc} +1 & +6 \\ +6 & +1 \\ +2 & +3 \\ +\mathbf{3} & +\mathbf{2} \end{array}} \right\} L = 6; \text{ need both " + " because of } +11x$$

Method 2: Factoring by Grouping

4 and 6 are neither prime nor "1," so we may prefer this method. We want two integers whose product is $ac = (4)(6) = 24$ and whose sum is $b = 11$. We want 8 and 3; split the middle term accordingly.

$$\begin{aligned} 4x^2 + 11x + 6 &= 4x^2 + \underbrace{8x + 3x}_{\text{OK to switch}} + 6 \\ &= (4x^2 + 8x) + (3x + 6) \quad \leftarrow \text{Group terms} \\ &= 4x(x + 2) + 3(x + 2) \quad \leftarrow \text{"Local factoring"} \\ &= (4x + 3)(x + 2) \quad \leftarrow \text{"Global factoring"} \end{aligned}$$

f) $2x^2 + 10x + 5$ is **prime** or **irreducible over the integers** (i.e., it cannot be broken down further using integer coefficients). None of these combinations work:

$$\begin{array}{cc} (2x &) \\ +1 & +5 \\ +5 & +1 \end{array} (x &) \leftarrow F = 2x^2; \text{ factors must be } 2x \text{ and } x$$

↓ Need $L = 5$; need both " + " because of $+10x$

We could also apply the **Test for Factorability**. The **discriminant**

$b^2 - 4ac = (10)^2 - 4(2)(5) = 100 - 40 = 60$, which is **not** a perfect square, and the GCF = 1, so the polynomial is **prime**.

g)

$$-3x^2 + 6x - 3 = \underbrace{-3}_{\text{GCF}} \underbrace{(x^2 - 2x + 1)}_{\text{a PST}}$$

You should usually factor out the GCF first.

$$= -3(x - 1)^2$$

- h) Apply the Difference of Two Squares formula $[a^2 - b^2 = (a + b)(a - b)]$ twice:

$$\begin{aligned} \underbrace{x^4}_{(x^2)^2} - \underbrace{16}_{(4)^2} &= \underbrace{(x^2 + 4)}_{\text{prime}} \left(\underbrace{x^2}_{(x)^2} - \underbrace{4}_{(2)^2} \right) \\ &= (x^2 + 4)(x + 2)(x - 2) \end{aligned}$$

- i) Use Factoring by Grouping:

$$\begin{aligned} a^3 - 3a + 2a^2b - 6b &= (a^3 - 3a) + (2a^2b - 6b) \\ &= a(a^2 - 3) + 2b(a^2 - 3) \\ &= (a + 2b)(a^2 - 3) \end{aligned}$$

- j) $4x^2 + 9y^2$ is **prime**. The GCF = 1, and we have no formula for the Sum of Two Squares (for now...; this will change when we discuss imaginary numbers in Section 2.1).

- k) Apply the Sum of Two Cubes formula $a^3 + b^3 = \underbrace{(a + b)}_{\text{"Expected factor"}} \left(\underbrace{a^2 - ab + b^2}_{\substack{\text{NOT} \\ -2ab}} \right)$:
- The visible signs follow the pattern: same, different, "+"

$$\underbrace{x^3}_{(x)^3} + \underbrace{125y^3}_{(5y)^3} = (x + 5y)(x^2 - 5xy + 25y^2)$$

- l) Apply the Difference of Two Cubes formula $a^3 - b^3 = \underbrace{(a - b)}_{\text{"Expected factor"}} \left(\underbrace{a^2 + ab + b^2}_{\substack{\text{NOT} \\ +2ab}} \right)$:
- The visible signs follow the pattern: same, different, "+"

$$\underbrace{x^3}_{(x)^3} - \underbrace{125y^3}_{(5y)^3} = (x - 5y)(x^2 + 5xy + 25y^2). \S$$