CH. 2

2.1: LIMITS and CONTINUITY

A Functions (1.3)

Ex Let f be the function defined by the rule $f(x) = \dfrac{x^2-16}{x-4}$

How do we eval?

A fraction bar automically groups $\frac{N}{D}$

1 $\underbrace{f(3)}_{\text{"f of 3" NOT "f times 3"}} = \dfrac{(3)^2-16}{(3)-4}$ ← work out ← before ÷

$= \dfrac{-7}{-1}$

$= 7$

$\boxed{f(3) = 7}$

cigarette

Input 3 ⇒ $\boxed{\text{f button}}$ ⇒ Output 7

2 $f(3.9) = \dfrac{(3.9)^2-16}{(3.9)-4}$

$\boxed{f(3.9) = 7.9}$

3.9 ⇒ $\boxed{f}$ ⇒ 7.9

(3) $f(4) = \dfrac{(4)^2-16}{(4)-4}$ ← ÷ by 0 is "illegal"

$\boxed{f(4) \text{ is undefined}}$

Carly's calc.
the finger

$4 \Rightarrow \boxed{f} \Rightarrow$ "ERROR!"

$f(x) = \dfrac{x^2-16}{x-4}$ is defined for

$\{x \mid x \neq 4\}$.

the set of all real values "x" | such that | x is not 4

what do you call the set of legal inputs?
Corresp. outputs?
Range(x^2): nonneg.
We never have a calc. button $\boxed{f}$

This is the [natural] domain of f.

Input in nat. dom. $\Longrightarrow$ $\boxed{\text{func. } f}$ $\Longrightarrow$ Exactly 1 real output (in the range of f)

Table of f values

x	f(x)	
3	7	⇒ (3,7) on graph
3.9	7.9	
4	und.	⇒ No point (literally)

IQ Test
If I input 3.99
we guess...
Let's confirm

Nice trick: Simplify 1st!

$$f(x) = \frac{x^2-16}{x-4} \leftarrow \text{Factor} \quad , x \neq 4$$

$$= \frac{(x+4)\cancel{(x-4)}}{\cancel{x-4}} \quad , x \neq 4$$

We can cancel (the $(x-4)$ factors) if $x \neq 4$.

$$= x+4 \quad , x \neq 4$$

gives simple yet complete descrip. of f

$$f(x) = \frac{x^2-16}{x-4} = \begin{cases} x+4 & , x \neq 4 \\ \text{und.} & , x = 4 \end{cases}$$

Table Easy!

x	f(x)		Graph
3	3+4=7	⇒	(3, 7)
3.9	3.9+4=7.9	⇒	(3.9, 7.9)
3.99	7.99	⇒	(3.99, 7.99)
4	und.	⇒	No point

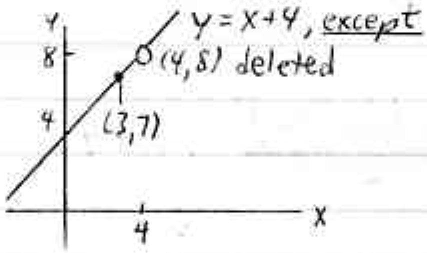
literally

Why?

Graph $y = f(x)$

independent variable

dependent



(B) Limits

Old Ex $\quad f(x) = \dfrac{x^2-16}{x-4}$

We can't say "$f(4) = 8$."

We should be able to say SOMETHING...

We can say

$$\text{“}\lim_{x \to 4} f(x) = 8.\text{”}$$

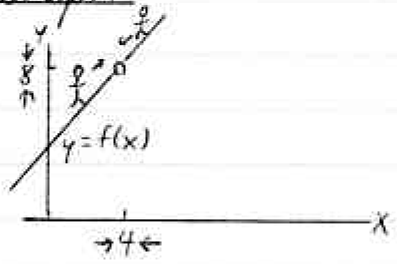
i.e., "The limit of $f(x)$ as x approaches 4 is 8."

i.e., "$f(x) \to 8$ as $x \to 4$."

Ways to see this

① Graph

skinny lovers but alas can never meet
Bill + Hillary
Siegfried & Roy

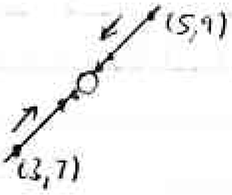


We'll talk more about this later.

② Tables

x	f(x)
3	7
3.9	7.9
3.99	7.99
as ↓	then ↓
4	8
↑	↑
4.01	8.01
4.1	8.1
5	9

x	f(x)
5	9
4.1	8.1
4.01	8.01
↓	↓
4	8



③ Simplifying 1st

$$\lim_{x \to 4} \frac{x^2-16}{x-4} = \lim_{x \to 4} \frac{(x+4)\cancel{(x-4)}}{\cancel{x-4}}$$

☆

$$= \lim_{x \to 4} (x+4)$$

We can: see E

Plug in $x = 4$.

$= 4 + 4$

$= \boxed{8}$

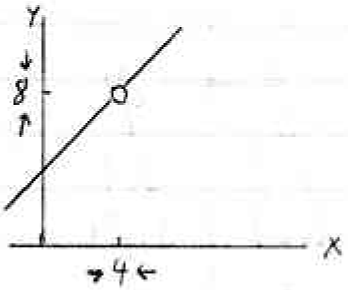
A Why is this OK?

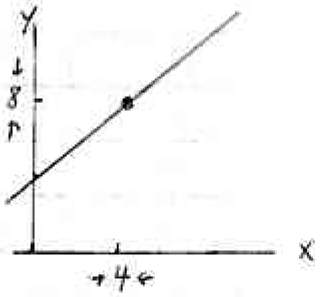
$y = \dfrac{x^2 - 16}{x - 4}$ $\qquad$ $y = x + 4$

Except here, the lines meet.

You're filling in the dot.

lim doesn't care.





In both cases, as $x \to 4$, then $y \to 8$.
What happens at $x = 4$ is technically irrelevant!

Up to 27
2nd: 25

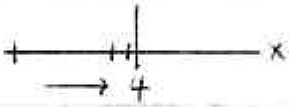
© One-Sided Limits

Old Ex $f(x) = \dfrac{x^2-16}{x-4}$

Table 1

x	$f(x)$
3	7
3.9	7.9
3.99	7.99
↓	↓
4	8

As $x \to 4$ from the left (from lower values),
then $f(x) \to 8$.



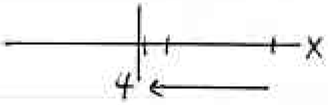
$$\boxed{\lim_{x \to 4^-} f(x) = 8}$$

from the left

Table 2

x	$f(x)$
5	9
4.1	8.1
4.01	8.01
↓	↓
4	8

As $x \to 4$ from the right (from higher values), then $f(x) \to 8$.



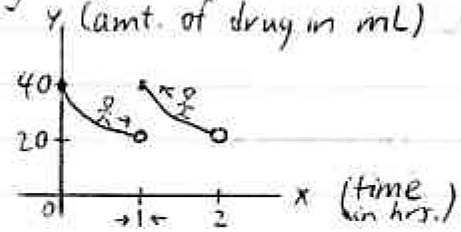
$$\boxed{\lim_{x \to 4^+} f(x) = 8}$$ ☆☆

from the rt.

☆ = ☆☆, so $\boxed{\lim_{x \to 4} f(x) = 8}$ (Two-sided limit exists.)

Finally... Got your att'n

Ex (Drug Injections)



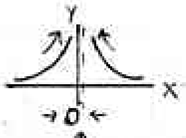
fix Dr. Banner, Kevork

$$\boxed{\lim_{x \to 1^-} f(x) = 20}$$
$$\boxed{\lim_{x \to 1^+} f(x) = 40}$$
$\neq$, so $\boxed{\lim_{x \to 1} f(x) \text{ does not exist (DNE)}}$

Up to 35
2nd: 33

(D) Limits and Infinity (∞)

Ex $f(x) = \frac{1}{x^2}$



(Can do Table)

x	$f(x)$
-1	1
-0.1	100
-0.01	10000
0	und.
0.01	$10{,}000$
0.1	100
1	1

Vertical Asymptote (VA)
a line the graph approaches

Is ∞ a real #?

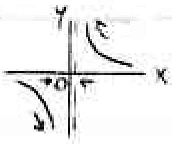
$\lim_{x \to 0^-} f(x) = \infty$
$\lim_{x \to 0^+} f(x) = \infty$

Match, so $\boxed{\lim_{x \to 0} f(x) = \infty}$

$\infty, -\infty$ are special cases of DNE.

Ex $f(x) = \frac{1}{x}$

Hell

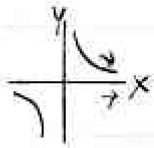


$\lim_{x \to 0^-} f(x) = -\infty$
$\lim_{x \to 0^+} f(x) = \infty$

Do not match, so $\lim_{x \to 0} f(x)$ DNE ($\infty, -\infty$ inappropriate)

up to 51 2nd:49

$\lim_{x \to \infty} f(x) = 0$



(E) Continuity

w/out...?

~ can trace graph w/out lifting pencil

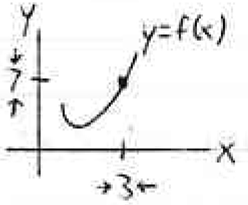
A func. f is <u>continuous</u> at c $(x=c)$

$\Longleftrightarrow$ if and only if

$$\begin{cases} \text{① } f(c) \text{ is defined} & \text{("There is a point.")} \\ \text{② } \lim_{x\to c} f(x) \text{ exists} & \\ \text{③ } \lim_{x\to c} f(x) = f(c) & \text{(①,② are =)} \end{cases}$$

Otherwise, f is <u>discontinuous</u> at c.

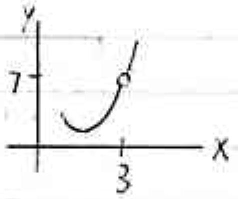
<u>Ex</u>



$$\left.\begin{array}{l} \text{① } f(3) = 7 \\ \text{② } \lim_{x\to 3} f(x) = 7 \\ \text{③: ① = ② ✓} \end{array}\right\} \text{, so } \boxed{f \text{ is cont. at } 3}$$

HW: Know how to read a problem 1,2,3

<u>Ex</u>

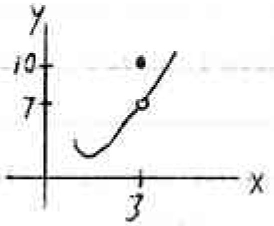


Is the lim 7?

If either ①,② FAILS, it's pointless to consider ③

① FAILS: $f(3)$ is und. $\rightarrow$ $\boxed{f \text{ is discont. at } 3}$

NOTE: ② OK: $\lim_{x\to 3} f(x) = 7$

Ex 

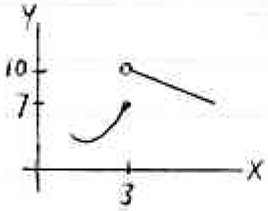
On test, ①,②,③

① OK $f(3) = 10$

② OK $\lim\limits_{x \to 3} f(x) = 7$

③ FAILS $\neq \Rightarrow$ $\boxed{f \text{ is discont. at } 3}$

on that day

Ex 

① OK: $f(3) = 7$, BUT

$\infty, -\infty$ also bad (DNE)

$\lim\limits_{x \to 3^-} f(x) = 7$

$\lim\limits_{x \to 3^+} f(x) = 10$

$\neq \Rightarrow$ $\lim\limits_{x \to 3} f(x)$ DNE ② FAILS

$\Downarrow$

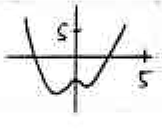
$\boxed{f \text{ is discont. at } 3}$

Up to 57
2nd: 57

f is <u>continuous [everywhere]</u> $\Leftrightarrow$
f is cont. at <u>all</u> real values

<u>Ex</u> Polynomial funcs. are cont. e-where.

<u>Ex</u> $f(x) = 2x^4 - 3x^2 + x - 4$



$\boxed{f \text{ is cont. e-where.}}$

More generally, <u>rational funcs.</u> are cont. on their domain.

$f(x)$ can be written as $\frac{\text{poly}}{\text{poly}}$ ← not 0

<u>Ex</u> $f(x) = \dfrac{3x^2-10}{x^3-x^2-2x}$ ← Factor

$$= \frac{3x^2-10}{x(x^2-x-2)}$$

$$= \frac{3x^2-10}{x(x+1)(x-2)}$$

If I have 3 #s, and I say product $\neq 0$

$f(x)$ is und. $\Leftrightarrow$ Denom. $= 0$

($\Leftrightarrow$ Any factor of the denom. $= 0$)

$\Leftrightarrow x = 0$ or $x+1=0$ or $x-2=0$

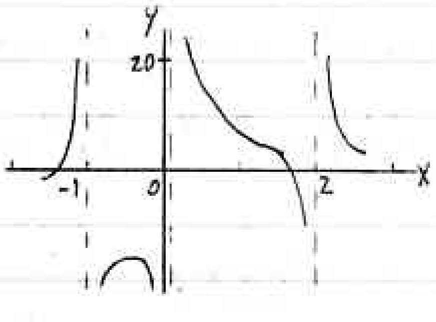
$x=-1$ $\quad x=2$

Domain $= \{x \mid x \neq 0,\ x \neq -1,\ x \neq 2\}$

Also, where f is cont.

Up to 70

$$\boxed{f \text{ is discont. at } x=0,\ x=-1,\ x=2}$$



Ⓕ Back to Limits

Circular reasoning

If f is cont. at c $\overset{\text{by def'n}}{\Longrightarrow}$ $\lim\limits_{x \to c} f(x) = f(c)$

(Direct sub works: Plug in $x=c$.)

$\frac{1}{x}$

What happens to graph $x \leftarrow x-1$

Asym at 1 instead of 0.

Ex $\lim\limits_{x \to 3} \frac{1}{x-1}$

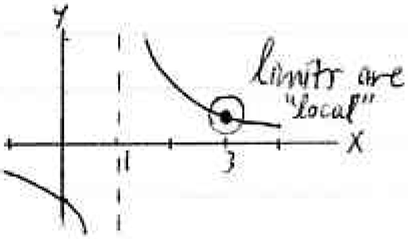
rat'l

cont. at 3

$\Rightarrow$ Plug in $x=3$

$= \frac{1}{(3)-1}$

$= \boxed{\frac{1}{2}}$



Here, it helps to know what happens at $x=3$.