2.4: PRODUCT and QUOTIENT RULES

A) Product Rule

$$(f+g)' = f' + g'$$

BUT $(fg)' \neq f'g'$ usually

How do you D_x a product?

$$D_x(fg) = f'g + fg'$$

For each term, D_x one factor, copy the others.

Grid
Shipping toxic waste along diagonal

$$D_x(fgh) = f'gh + fg'h + fgh'$$

etc.

Would've been fair on 1st test.
"Pointer Method"

Ex Find $D_x\left[\overbrace{(x^4+3)(3x^2-4)}^{\text{FOIL or ...}}\right]$ using Product Rule.

$\wedge(D_x)$ copy
$+$ copy $\wedge$

$$= (4x^3)(3x^2-4) + (x^4+3)(6x)$$

great if can stop here

Simplify:

$$= 12x^5 - 16x^3 + 6x^5 + 18x$$

(May help to line up like terms)

$$= \boxed{18x^5 - 16x^3 + 18x}$$

(B) Quotient Rule

$$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'} \quad \text{usually}$$

How to Ace: Hi D(Ho) chucky?

J-Lo

In fact,

$$\left(\frac{f}{g}\right)' = \frac{\text{Lo D(Hi)} - \text{Hi D(Lo)}}{\text{Square of what's below}} \quad \text{rhyme}$$

$$= \frac{gf' - fg'}{g^2}$$

Ex Find $D_x\left(\frac{6x+1}{2x^2-3}\right)$

$$= \frac{(2x^2-3)\cdot D_x(6x+1) - (6x+1)\cdot D_x(2x^2-3)}{(2x^2-3)^2}$$

You can cancel a factor of the entire N w/ ' D

$$= \frac{(2x^2-3)(6) - (6x+1)(4x)}{(2x^2-3)^2} \quad \leftarrow \text{Simplify}$$

Can't strike

Careful!

$$= \frac{12x^2 - 18 - (24x^2 + 4x)}{(2x^2-3)^2}$$

$$= \frac{12x^2 - 18 - 24x^2 - 4x}{(2x^2-3)^2}$$

Factor if you think it'll help cancellation!

$$= \boxed{\frac{-12x^2 - 4x - 18}{(2x^2-3)^2}}$$

Ex $D_x\left(\frac{4}{x^7}\right)$ Quot. Rule or

Turning it into an easy product.

$$= D_x(4x^{-7})$$
$$= -28x^{-8}$$
$$= \boxed{-\frac{28}{x^8}}$$

Ⓒ Marginals

2.3 Ex x = # dolls

Profit $P(x) = -2x^2 + 120x - 1000$ (in \$)

If I make a profit of \$86 off of 3 dolls, what's my A.P. per doll? Test score avg

$\Rightarrow$ Average Profit $AP(x) = \dfrac{P(x)}{x}$

$$= \frac{-2x^2 + 120x - 1000}{x}$$

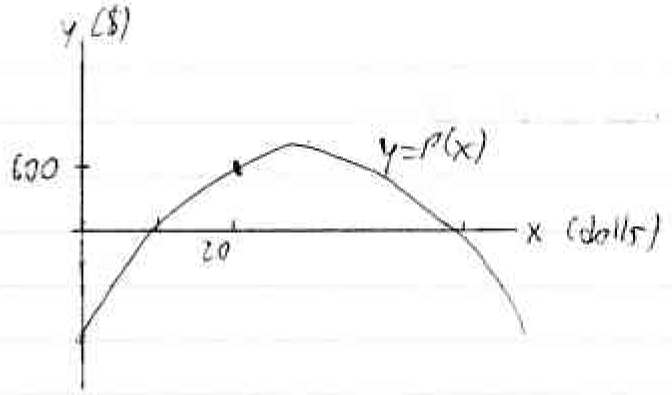
or $-2x + 120 - \frac{1000}{x}$

$\Rightarrow$ Marginal Average Profit $MAP(x)$

$$= D_x[AP(x)]$$

Quot. Rule or use

$$= D_x(-2x + 120 - 1000x^{-1})$$
$$= -2 + 1000x^{-2}$$
$$= -2 + \frac{1000}{x^2}$$



$$P(20) = -2(20)^2 + 120(20) - 1000$$
$$= 600 \text{ (\$)}$$

Total profit for 20 dolls is \$600.

$$AP(20) = \frac{600}{20}$$
$$= 30 \left(\frac{\$}{\text{doll}}\right)$$

Avg. profit for 20 dolls is $30 \frac{\$}{\text{doll}}$.

Explain "marginal" stuff carefully!

$$MAP(20) = -2 + \frac{1000}{(20)^2}$$
$$= 0.5$$

If go to 21 dolls, avg. profit ≈ \$30.50 per doll (gets worse at an approx. as # dolls ↗)

absolutely no good!

When 20 dolls have been produced and sold, the average profit increases by about 50¢ per doll for each add'l doll.

$MAR(x)$ Revenue

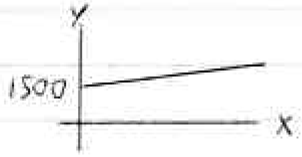
$MAC(x)$ Cost

2nd, 3rd Ed.
Machinery
Let's construct a simple cost func.

Ex 7 Fixed costs : \$1500.
Then, it costs \$12 to produce each Potter book
$x = \#$ books produced

what's the cost func?

Total cost $C(x) = 12x + 1500$ (in \$)

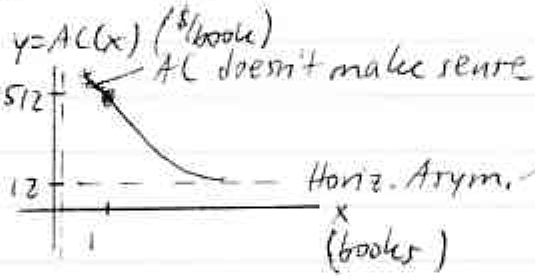


Total costs rise

Avg. cost $AC(x) = \dfrac{12x+1500}{x}$

$= 12 + \dfrac{1500}{x}$

$\lim_{x\to\infty}\left(12 + \dfrac{1500}{x}\right) = 12$



(1, 1512)
1500+12
$\frac{1524}{2} = 762 \frac{\$}{book}$

Avg. Cost ↘ cost of 1 more book

Addison-Wesley has easier time than Mom-Pop.

Effect of fixed costs dissipates w/higher production

Economies of mass production. (scale)

If truncate for all $x \geq 1$?

slopes: $MAC(x) < 0$ for all $x \geq 1$, because:

$$AC(x) = 12 + 1500x^{-1}$$

$$\Rightarrow MAC(x) = -1500x^{-2}$$

$$= -\frac{1500}{x^2} < 0 \text{ for all } x \geq 1$$

(2.5): HIGHER-ORDER DERIVS.

(A) Notation

$$f(x) = x^3 \overset{D_x}{\Rightarrow} f'(x) = 3x^2 \overset{D_x}{\Rightarrow} f''(x) = 6x$$

y	y'	y''
	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
	$\frac{d}{dx}(x^3)$	$\frac{d^2}{dx^2}(x^3)$
	$D_x(x^3)$	$D_x^2(x^3)$

HW prob
Idea: poly → 0, even 93'/2-deg.
Need ( )
o.w.
$f^2 = f \circ f$

$$\Rightarrow f'''(x) = 6$$
$$\Rightarrow f^{(4)}(x) = 0$$
$$\Rightarrow f^{(5)}(x) = 0$$

order

(B) Exs

Like 2nd, 3rd Ed 10

Ex

split

$$f(x) = \frac{x^2 - 6}{4x}$$

Find $f''(2)$. i.e., $\frac{d^2}{dx^2}\left(\frac{x^2-6}{4x}\right)\Big|_{x=2}$

$$f(x) = \frac{\cancel{x^2}^{\,x}}{\cancel{4x}_{\,1}} - \frac{\cancel{6}^{\,3}}{\cancel{4}_{\,2}x}$$ (Quot. Rule last resort.)

$$= \frac{x}{4} - \frac{3}{2x}$$

≈ poly

$$= \frac{1}{4}x - \frac{3}{2}x^{-1}$$

$$\Rightarrow f'(x) = \frac{1}{4} + \frac{3}{2}x^{-2}$$

$$\Rightarrow f''(x) = \frac{3}{2}\left(-\cancel{2}x^{-3}\right)$$
$$= -3x^{-3}$$
$$= -\frac{3}{x^3}$$

$$\Rightarrow f''(2) = -\frac{3}{(2)^3}$$

$$= \boxed{-\frac{3}{8}}$$

Do you think Quot. Rule good idea?

<u>WARNING</u>: If $f(x) = \dfrac{4x}{x^2 - 6}$ can't split!

$(\ )^{-1}$ need Chain

Use Quot. Rule.

Like #16

<u>Ex</u> $f(x) = \frac{32}{\sqrt[4]{x}}$. Find $f''(16)$

$$f(x) = 32x^{-1/4}$$

$$\Rightarrow f'(x) = 32\left(-\frac{1}{4}x^{\overbrace{-5/4}^{-\frac{1}{4}-\frac{4}{4}}}\right)$$
$$= -8x^{-5/4}$$

$$\Rightarrow f''(x) = \cancel{-}\cancel{8}^{2}\left(\cancel{-}\frac{5}{4}x^{\overbrace{-9/4}^{-\frac{5}{4}-\frac{4}{4}}}\right)$$
$$= 10x^{-9/4}$$
$$= \frac{10}{x^{9/4}}$$

Turns out $(\sqrt[4]{x})^9$ better than $\sqrt[4]{x^9}$

$$= \frac{10}{(\sqrt[4]{x})^9}$$

$$\Rightarrow f''(16) = \frac{10}{(\sqrt[4]{16})^9}$$

$$= \frac{10}{(2)^9}$$

Think computer

$$= \frac{10}{512}$$

Fair w/ last calculator.

$$= \boxed{\frac{5}{256}}$$

Go back one w/ power of 2. 3rd: Up to 25

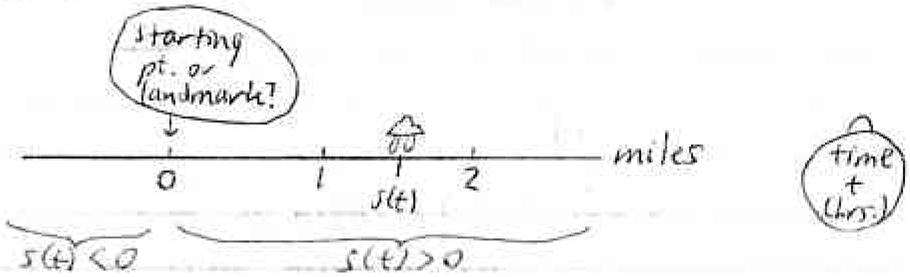
Can't have a calculus class without this...

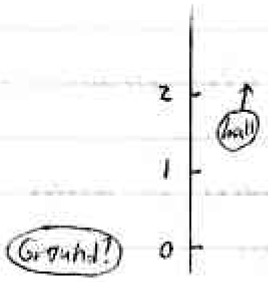
Ⓒ Physics

Car / Tank

Dave's house

Distance is 20





$s(t)$ = position (directed distance) [of car at time t]

What is rate of change of position? What do we call

$v(t) = s'(t) =$ velocity ... [of car at time t]

Inst. rate of change of position
w/respect to time.

① $|v(t)| =$ speed

Cank-serious stalker

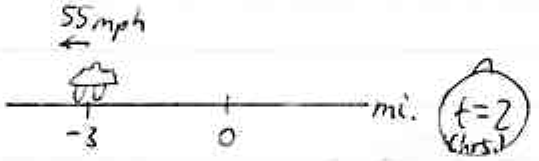
② $v(t)<0 \leftarrow$ $\rightarrow v(t)>0$

$\uparrow v(t)>0$
ball
$\downarrow v(t)<0$

Direction indicates sign of v.

Ex

← police car



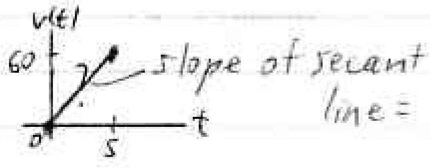
People had difficulty w/these.

$s(2) = -3$ (mi.)
$v(2) = -55$ (mph, or $\frac{mi.}{hr.}$)
or 55 mph "west" or "left"

$a'(t) =$ jerk

$a(t) = v'(t) = s''(t) =$ acceleration ... [of car at time t]

Ex A car speeds up from 0 mph to 60 mph in 5 sec.



How many secs in an hr.?

rate of change of vel wrt time

$$\text{Avg. acc.} = \frac{60-0}{5}$$
$$= \frac{12\ \frac{mi.}{hr.}}{sec.} \times \frac{3600\ sec}{1\ hr.}$$
$$= 43{,}200\ \frac{mi.}{hr.^2}\ \text{(mi. per hr. per hr.)}$$

<u>If</u> constant acc. $\Rightarrow$
0 mph to 43,200 mph in 1 hr.

Earth

 $C \approx 25{,}000$ mi.

In section run
If driving on
straight line
to east

<u>If</u> always (for $t \geq 0$)

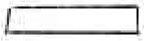
0

What
measures
the
deriv.?

Some of
you
whoa!

Up to 39 (2nd)

Odometer



measures $s(t)$

Speedometer

$v(t) = s'(t)$

Computations

See Ex 5.

from 2nd Ed/CDC
Applic. Preview
#48 in 3rd, p.153
: CDC 1981-2000
1 thru ~~on~~ in Jan. 1

<u>1 AIDS Ex</u>

x = # years after Jan. 1, 1990 $(0 \leq x \leq 6)$

"cumulative"
: incl. dead?

$f(x) = 0.13x^4 - 2.6x^3 + 13.1x^2 + 51x + 200$
= total # of AIDS cases (in $\approx$1000s of cases)

How do I find #
cases on 1/1/95?

$f(5) = 538.75$

$\boxed{\text{On Jan. 1, 1995, there were about 538,750 AIDS cases.}}$

Were there
$\approx$538,539 cases?

What does this tell me?

$$f'(x) = 0.52x^3 - 7.8x^2 + 26.2x + 51$$
$$f'(5) = 52$$

Let's compare w/ what f gives
$f(6) = 584.48$
≈ Marginal idea

There are actually fewer cases!

On Jan. 1, 1995, AIDS cases were increasing by about 52,000 [new] cases per year.

Can skip (On Jan. 1, 1996, we expect ≈ 538,750 + 52,000 = 590,750 cases)
(*Note* $f(6) \approx 584.48 \Rightarrow \approx 584{,}480$ cases, actually.)

$$f''(x) = 1.56x^2 - 15.6x + 26.2$$
$$f''(5) = -12.8$$

On Jan. 1, 1995, the rate of increase in AIDS cases is decreasing by about 12,800 cases per year per year.

f'': rate of change of rate of change of f

Prevention?

1/1/95 : AIDS ↗ by ≈ 52,000 cases/yr.
↓ −12,800
1/1/96 : AIDS ↗ by ≈ 39,200 cases/yr. (estimate)

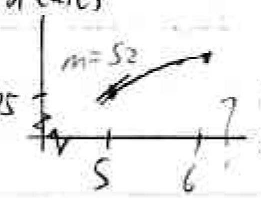
Rate of growth on 1/1/96. Project → 1/1/97?

<u>Note</u> $f'(6) = 39.72 \rightarrow \approx 39{,}720$ cases/year, actually.
"one-sided"

Probably nothing dramatic on 1/1/96

Looks linear
IP ≈ 2.1, ~~2x~~ at

<u>Idea</u>



1st deriv. > 0
2nd < 0

<u>Note</u>

1st deriv. > 0
2nd > 0

In book: Nixon "The rate of increase of inflation is decreasing." Fall '72. 3rd deriv.

cases are increasing, but at a decreasing rate.

Annual costs are increasing, and at an increasing rate.

(2.6): CHAIN RULE

(A) Idea

Dick runs twice as fast as Harry.
Tom runs 3x " " Dick.
⇒ Tom runs 6x " " Harry.

Actually, Tom's distance covered

$$\frac{d(\text{Tom})}{d(\text{Harry})} = \frac{d(\text{Tom})}{d(\text{Dick})} \cdot \frac{d(\text{Dick})}{d(\text{Harry})}$$

$$6 = 3 \cdot 2$$

p.172 (2nd ed.)

(B) In Leibniz's Notation

f of u please

Chain: composition of funcs.

$$\left.\begin{array}{c} y \\ \mid \; y = f(u) \\ u \\ \mid \; u = g(x) \\ x \end{array}\right\} y = f(g(x))$$

Products of Ds come from comps. of funcs., not products of funcs.

Looks like "du"s cancel, but not technically the case!

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\overbrace{\frac{du}{dx}}^{\text{tail}}}$$

if exist

Deriv of outside · Deriv of inside

i.e., $y' = f'(u) \cdot u'$ (Deriv. of outside · Deriv. of inside)

D_x to get tail

(C) Ex Find $\frac{d}{dx} \underbrace{(3x^2+4)^7}_{=y}$

composite func.

where

$u = 3x^2+4$ "inside"

$y = u^7$

What do we do to u to get y? 2 calc. buttons ideally, $\boxed{3x^2+4}$ $\boxed{u^7}$

" Boss gives you 7" "Hey, what's u?"

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= (7u^6)(6x)$$

unwrap

$$= 7(3x^2+4)^6(6x)$$

$$= \boxed{42x(3x^2+4)^6}$$

We study applic. of Chain Rule

(D) Generalized Power Rule

$$D_x(x^n) = nx^{n-1}$$

$$\boxed{D_x(u^n) = nu^{n-1} \cdot \overbrace{D_x u}^{\text{tail}}}$$

func. of x

$\frac{dy}{du}$: like basic power rule

Comes from Chain Rule: $y = u^n$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Old Ex $D_x(3x^2+4)^7 = 7(3x^2+4)^6 \cdot \overbrace{(6x)}^{\text{tail}}$

D_x

$$= \boxed{42x(3x^2+4)^6}$$

Ex $f(x) = \dfrac{1}{\sqrt{2x^3 - x}}$.
Find $f'(x)$.

$$f(x) = \frac{1}{(2x^3-x)^{1/2}}$$

Quot. Rule or

$$= (2x^3-x)^{-1/2}$$

$$\Rightarrow f'(x) = -\tfrac{1}{2}(2x^3-x)^{\overbrace{-\frac{3}{2}}^{-\frac{1}{2}-\frac{2}{2}}} \cdot \underbrace{D_x(2x^3-x)}_{\text{tail } = (6x^2-1)}$$

$$= \boxed{-\tfrac{1}{2}(2x^3-x)^{-3/2}.(6x^2-1) \quad \text{or} \quad -\frac{6x^2-1}{2(2x^3-x)^{3/2}} \quad \text{or} \quad \frac{1-6x^2}{2(2x^3-x)^{3/2}}}$$

What's the opp of $a-b$? $b-a$.

Don't distribute!

Ex $f(x) = \underbrace{4x}_{\text{can treat as one factor}}(x^3+2)^5$

$f'(x) =$ ∧ copy
$+$ copy ∧

$$= (4)(x^3+2)^5 + (4x) \cdot 5(x^3+2)^4 \cdot \underbrace{(3x^2)}_{\text{tail}}$$

Can factor out $4(x^3+2)^4$

$$= \boxed{4(x^3+2)^5 + 60x^3(x^3+2)^4}$$

(E) HW #57 $^{2nd, 3rd}$ ed.

P is a func of x, which, in turn, is a func of t.

P : CO pollution

$$P(x) = 0.02x^{3/2} + 1 \text{ (ppm)}$$

You have more pollution if you have more ... in the city.

x : pop. of the city

$$x(t) = 12 + 2t \text{ (1000s of people)}$$

t : time (years from now)

(☆)

(☆) $\underbrace{P(x(t))}_{\text{Book calls this } P(t).} = 0.02(12+2t)^{3/2} + 1$ (ppm)

Find $P'(2)$, or $\left.\frac{dP}{dt}\right|_{t=2}$.

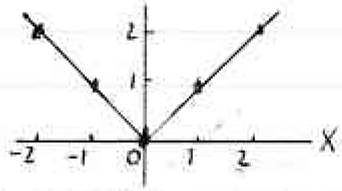
The rate that CO increases, w/respect to time 2 years from now.

Mr. Burns

$
|
x
|
t

2.7 : WHERE IS f' DNE?

A $f(x) = |x|$



A $|-2| = 2$

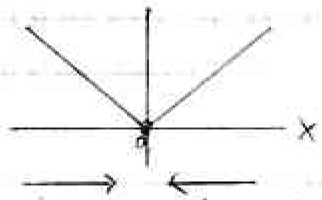
sign flip

If $x < 0$ $\Rightarrow f(x) = |x| = -x$ A $\Rightarrow f'(x) = -1$

If $x > 0$ $\Rightarrow f(x) = |x| = x$ $\Rightarrow f'(x) = 1$

What's $|7|$? $|-7|$?
$|M|$? $|-M|$?
$|x|$? $|x|$?
if $x>0$ if $x<0$

What is $f'(0)$?



Left-hand deriv. (m_L) at 0 is -1

Right-hand deriv. (m_R) is 1.

$m_L \neq m_R \Rightarrow \boxed{f'(0) \text{ DNE}}$

i.e., f is not differentiable at 0.

vs.
secant lines
→tan line
more appropriate
than
lim (tan line slopes)
(Not implied by ↗
Counterex:
Gelbaum 36
diff'e func w/
discont. deriv.
see BO L3-66

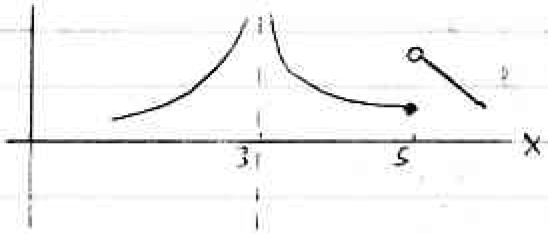
$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(B) Where is f' DNE?

(1) Wherever f is not cont.

Ex



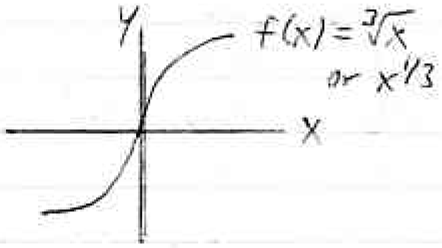
(f is not differentiable at 3 and 5.)
$f'(3)$ and $f'(5)$ DNE.

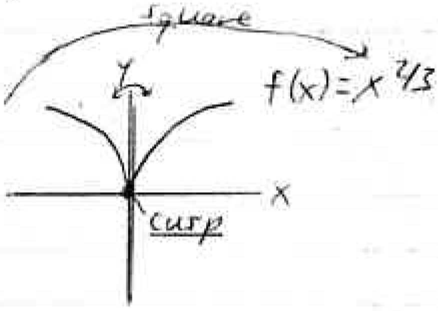
If you look at the graph of f, deriv are ___ What kind of line has und. slope?

(2) Wherever there is a vertical tangent line.
(undefined slope)

$x^{2/3}$ never neg. even sym about y axis

Exs



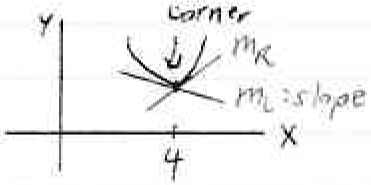


$f'(0)$ DNE

$f'(0)$ DNE

(3) Wherever $m_L \neq m_R$.

Graph changes smoothly
Tangent line changes abruptly 2 at pt.



$f'(4)$ DNE

REVIEW 2.4 - 2.7 (OUTLINE)

D_x TECHNIQUES

2.4 PRODUCT RULE

^ Method

$$(fg)' = f'g + fg'$$

QUOTIENT RULE

$$\left(\frac{f}{g}\right)' = \frac{\text{Lo } D(\text{Hi}) - \text{Hi } D(\text{Lo})}{\text{Square of what's below}}$$ rhyme

$$= \frac{gf' - fg'}{g^2}$$

Last resort?

2.6 CHAIN RULE

Composite funcs. p, x, t : $P(x(t))$

Gen. Power Rule

$$D_x(u^n) = nu^{n-1} \cdot \underbrace{D_x u}_{\text{tail from Chain Rule}}$$

Combining rules

(2.4) BUSINESS

Average Profit $AP(x) = \frac{P(x)}{x}$
Marginal Average Profit $MAP(x) = D_x[AP(x)]$

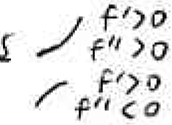
Also: $\overset{\text{Cost}}{AC(x)}$, $\overset{\text{Revenue}}{AR(x)}$
$MAC(x)$, $MAR(x)$

Interpret answers
Consider $\lim_{x \to \infty}$

(2.5) $f', f'', f''', \ldots$

Word probs.: Interpret answers
Units

Figure out signs of f', f'' from a graph Exs



Notation
Ex If $y = f(x)$, $f''(8) = \frac{d^2 f}{dx^2}\Big|_{x=8}$ (or y)

position, velocity, acceleration (Units!)

position	velocity	acceleration
s	s'	s''
	v	v'
		a

(2.7) WHERE IS f' DNE?

not continuous
vertical tangent line
$m_L \neq m_R$