(3.6): IMPLICIT DIFFERENTIATION (IMP. DIFF.) and RELATED RATES

(A) When does Imp. Diff. help?

Ex Find y' (or $\frac{dy}{dx}$) if $y-x^2=0$.

$$y-x^2=0$$
defines y implicitly as a func. of x

$\overset{\text{equiv.}}{\Longleftrightarrow}$ $y=x^2$, explicitly
Form: $y=f(x)$

$$\boxed{y'=2x}$$

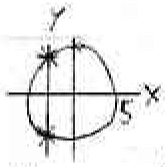

can still talk about slopes of tan lines

Ex Find y' if $y^3+y+x=4$.

Hard to solve for y!

(1) Imp. Diff. helps when it's hard to get the form $y=f(x)$.

Ex $x^2+y^2=25$



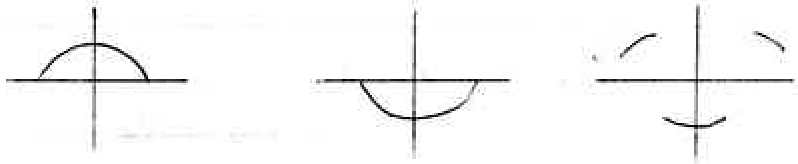
hits more than 1 pt.

fails VLT

② Imp. Diff. helps when the eq. does <u>not</u> determine y as a func. of x.

However, the eq. determines many <u>implicit funcs.</u> of x
$y = f(x)$
take any piece(s) of the graph that pass VLT.

Smiley face

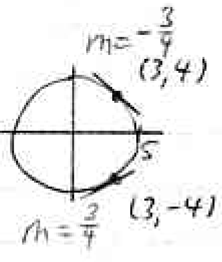


Imp. Diff. deals w/these simultaneously.

<u>Ex</u> $x^2+y^2=25$ (2nd Ed pp. 260-2, 3rd Ed pp. 248-50)

Can't talk about $f'(3)$

Does it make sense that this slope is "+" / "−"?



Find the slope of the tan line at (3,4) and (3,-4).

Turns out $\frac{dy}{dx} = -\frac{x}{y}$ (depends on y!!)

$$\Rightarrow \begin{cases} \left.\frac{dy}{dx}\right|_{(3,4)} = \boxed{-\frac{3}{4}} \\ \left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4} = \boxed{\frac{3}{4}} \end{cases}$$

(Tiebreaker - we need to distinguish between (3,4), (3,-4))

What is
Let's be careful

<u>Note</u> $\left.\frac{dy}{dx}\right|_{(1,1)}$ is undefined, bec. (1,1) is <u>not</u> on the circle!

(B) Imp. Diff.

Assume $y = f(x)$ (implicitly).

Notation $D_x y = y'$ (or $\frac{dy}{dx}$)

Go slowly!

Exs (1) $D_x(7y) = 7D_x(y) = 7y'$ (Constant factors pop out.)

How do you D_x a square using Gen. Power Rule?
y': Deriv of base

(2) $D_x(\underset{f(x)}{y^2}) = 2y\underbrace{y'}_{\text{tail}=\frac{dy}{dx}}$

Ex $D_x[(\overbrace{x^3+1}^{=y})^2] = \underbrace{2(x^3+1)}_{y} \cdot \underbrace{3x^2}_{y'}$

(3) $D_x(y^3) = 3y^2y'$

(4) $D_x(xy) = D_x(x)\cdot(y) + (x)\cdot D_x(y)$
$\wedge$ copy
copy $\wedge$
$= (1)(y) + (x)(y')$
$= y + xy'$

Ex $x^2+y^2=25$. Find $\frac{dy}{dx}$, or y'.

D_x each term

$D_x(x^2) + D_x(y^2) = D_x(25)$

many say "25" - NO!

$2x + 2yy' = 0$

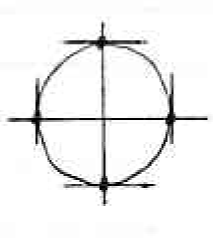
Solve for y'

$2yy' = -2x$

$y' = -\frac{2x}{2y}$

$\boxed{y' = -\frac{x}{y}}$

Look familiar?
$y'=0$ when $x=0$
y' is und. when $y=0$



Up to 13 (2nd)(3rd)

Ex $4x^3 + x^2y^3 = y^2$. Find y'.

① D_x each term

$$D_x(4x^3) + D_x(x^2y^3) = D_x(y^2)$$

+ copy   copy

$$12x^2 + (2x)(y^3) + (x^2)(3y^2y') = 2yy'$$

$$12x^2 + 2xy^3 + 3x^2y^2y' = 2yy'$$

② Isolate terms w/ y' on one side.

$$3x^2y^2y' - 2yy' = -12x^2 - 2xy^3$$

③ Factor out y'

$$y'(3x^2y^2 - 2y) = -12x^2 - 2xy^3$$

④ Isolate y' by $\div$

Up to 27 [illegible]

$$\boxed{y' = \frac{-12x^2 - 2xy^3}{3x^2y^2 - 2y}}$$

Let's get to business, literally

© Demand Eqs.

<u>Ex</u> (#38 2nd 3rd)

IN BOOK!

I'm adding →

A company's demand eq. is $x = \sqrt{650 - p^2}$, where p is the (unit) price in \$, and x is the qty. consumers will demand at that price.

Can do but painful →

Find $\frac{dp}{dx}$ when $p = 5$, and interpret!

OK: $x, p \geq 0$ assumed
No need to ✓: we eval later, anyway
Not recipr in general. Recipr at corresp pts.
If $y = x^3$ (OK, it's 1-1)
$\left.\frac{dy}{dx}\right|_{x=2} = \left.\frac{dx}{dy}\right|_{y=8}$

$$x = \sqrt{650 - p^2}$$

Find $\frac{dp}{dx}$, not $\frac{dx}{dp}$.

$$\Rightarrow x^2 = 650 - p^2 \quad (x \geq 0) \text{ (Easier)}$$

D_x each term

$$D_x(x^2) = \underbrace{D_x(650)}_{=0} - D_x(p^2)$$

$$2x = -2p \underbrace{\frac{dp}{dx}}_{\text{or } p'}$$

$$\frac{2x}{-2p} = \frac{dp}{dx}$$

$$\frac{dp}{dx} = -\frac{x}{p}$$

How find x when $p = 5$? Sometimes have to plug into eq.

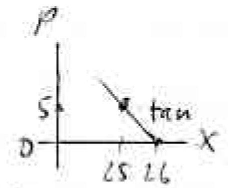
$$\text{When } p = 5 \Rightarrow x = \sqrt{650 - (5)^2}$$
$$= 25$$

vs. finding p 1st

$p(26) - p(25)$ $\quad$ $\left.\frac{dp}{dx}\right|_{(5,25)}$

$$\left.\frac{dp}{dx}\right|_{\substack{p=5\\x=25}} = -\frac{25}{5} = \boxed{-5}$$

Up to 37



(like a marginal analysis)

> When the unit price is \$5, the price must drop by about \$5 (+tree) if one more unit is to be demanded.

ⓓ Related Rates

IN BOOK!

Don't know $x(t)$ $x(t) = 20t + ?$

Ex (#52 2nd 3rd)

A company's profit from selling x units of an item is $P = 1000x - \frac{1}{2}x^2$ (\$). If sales are growing at the rate of 20 per day, find how rapidly profit is growing (in \$ per day) when 600 units have been sold.

(selling 20 units/day)

$$P = 1000x - \frac{1}{2}x^2$$

total # units sold thus far

Let t = time in days, x = sales, P = Profit
Given: $\frac{dx}{dt} = 20$ $\left(\frac{\text{units}}{\text{day}}\right)$
Find: $\frac{dP}{dt}$ when $x = 600$
$\Rightarrow D_t$ each term

20 is $\frac{d(\text{what})}{d(\text{what})}$

D each term wrt?

$$D_t(P) = D_t(1000x) - D_t\left(\tfrac{1}{2}x^2\right)$$

$$\frac{dP}{dt} = 1000\frac{dx}{dt} - x\underbrace{\frac{dx}{dt}}_{\text{tail}}$$

This eq. <u>relates</u> the <u>rates</u> $\frac{dP}{dt}$, $\frac{dx}{dt}$.

Avoid using P', x': too ambiguous.

Plug in: $\frac{dx}{dt} = 20$, $x = 600$ (only <u>after</u> you D_t)

for 1 day:
Avg. Profit $= \frac{8000}{20} = 400 \frac{\$}{\text{unit}}$?
If basic assumps. hold.
Maybe P more complex.
If $\frac{dx}{dt} = 20$ all the time since $x=0$
$x = 600$ when $t = 30$. Don't care about t values.
Maybe $x = 500$ (bulk) at $t = 0$, though.

$$\frac{dP}{dt} = 1000(20) - (600)(20)$$

$$\frac{dP}{dt} = \boxed{8000 \left(\frac{\$}{\text{day}}\right)}$$

<u>Interpret</u>

> Profit is growing at \$8000 per day when 600 units have been sold.