In terms of graphs f_x are slopes

(5.3): DEFINITE $\int$s and AREAS

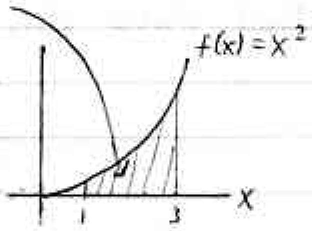
(A) Intro

Ex $\int_{1}^{3} x^2\,dx$ = the definite integral of x^2 from 1 to 3

upper limit

lower limit

= Area



bec. $x^2 \geq 0$ on $[1,3]$

$\overset{\text{turns out}}{=} 8\frac{2}{3}$ or $8.\bar{6}$

what do you think
Dummy var,
→replace him
w/another dummy,
nothing changes;
like airport
security

Ex $\int_1^3 t^2\,dt = 8.\bar{6}$, also

In Ch. 5,

An indefinite $\int$ gives a family of funcs.

Ex $\int x^2\,dx = \frac{x^3}{3} + C$

gives us...

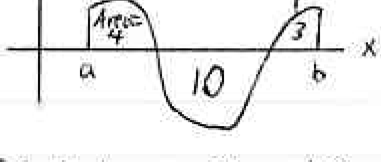
A definite $\int$ gives a #.

Ex $\int_1^3 x^2\,dx = 8.\bar{6}$

What if $f < 0$ sometimes?

What do you think
we do w/10
contribute
a neg. # to
the def. $\int$

Ex

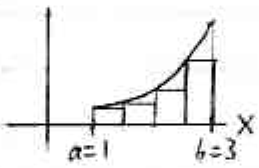


$\int_a^b f(x)\,dx = 4 - 10 + 3$
$= -3$

(B) Approximating Areas, Definite $\int$s Using Riemann Sums ($f \geq 0$)

Ex Approx. $\int_1^3 x^2\,dx$ using a Left Riemann Sum with $n=4$ rectangles of the same "width."
(i.e., Approx. the area under the curve $f(x)=x^2$ from $a=1$ to $b=3$.)

Step 1: Find Δx (delta), the rect. width.



n=4 rects.

width of interval $= b-a$ (= 2, here)

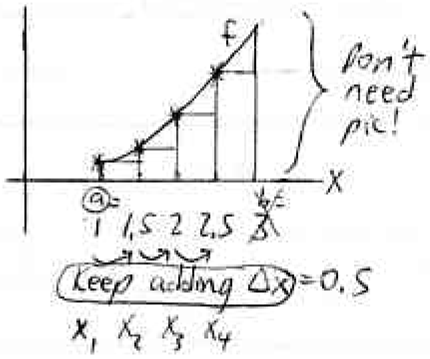
width of each rect. $= \Delta x = \frac{b-a}{n}$

Here, $\Delta x = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$ or 0.5

Step 2: Find the left breakpoints $(x_1, x_2, \ldots, x_n)$

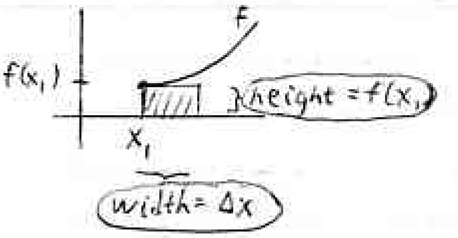
In a Left Riemann Sum,
the left corners of each rect.
lie on the curve and the x-axis.

x_i: where do we start/gaze? fenceposts



Note What is the area of the 1^{st} rect.?

If this is x_1, what's this y-coord?



$$\left(\begin{array}{l} \text{func. value at } x_1 \\ = \text{y-coord. of pt. (at } x_1) \\ = \text{height of rect.} \end{array} \right)$$

what's the area of the 2nd

$$\underset{\text{of 1st rect.}}{\text{Area}} = f(x_1) \cdot \Delta x$$

Step 3: Find the Left Riemann Sum

$= $ sum of the areas of the rects.

$= \underbrace{f(x_1)\cdot\Delta x}_{1st} + \underbrace{f(x_2)\cdot\Delta x}_{2nd} + \underbrace{f(x_3)\cdot\Delta x}_{3rd} + \underbrace{f(x_4)\cdot\Delta x}_{4th}$

$= [f(1)][0.5] + [f(1.5)][0.5] + [f(2)][0.5] + [f(2.5)][0.5]$

$f(x) = x^2$

$= [(1)^2][0.5] + [(1.5)^2][0.5] + [(2)^2][0.5] + [(2.5)^2][0.5]$

$= 0.5 + 1.125 + 2 + 3.125$

$= \boxed{6.75}$

2nd: Do 1, 3

How come we have an underest. of the true value?

Book: All Δx_i =

Area $\approx$ 6.75 square units

$\int_1^3 x^2\,dx \approx 6.75$ (Exact: $8.\overline{6}$)

n = # rects. (assume same width)

As $n \to \infty$,
Approxs. $\to$ Exact

> If f cont. on [a, b],
>
> $$\int_a^b f(x)\,dx = \lim_{n\to\infty} [\text{Left Riemann Sums}]$$

(Def'n)

Ⓒ An Easier Way to Find $\int_a^b f(x)\,dx$: Fundamental Thm. of Calculus (FTC)

> If f cont. on [a, b],
>
> $$\int_a^b f(x)\,dx = [\underset{\text{Find an AD of } f}{F(x)}]_a^b$$
>
> $$= \underbrace{F(b)}_{\text{Eval at } b} - \underbrace{F(a)}_{\text{Eval at } a}$$

Ex $\int_1^3 \underbrace{x^2}_{\text{cont. on } [1,3] \checkmark} dx = \left[\underbrace{\frac{x^3}{3}}_{\text{an AD of } x^2}\right]_1^3$ Don't need "+C"!!

$$= \underbrace{\left[\frac{(3)^3}{3}\right]}_{\text{(Eval. at 3)}} - \underbrace{\left[\frac{(1)^3}{3}\right]}_{\text{(Eval. at 1)}}$$

$$= 9 - \frac{1}{3}$$

$$= \boxed{8\tfrac{2}{3} \text{ or } 8.\overline{6}}$$

<u>Ex</u> Find the area under $f(x) = \underbrace{2 + \frac{1}{\sqrt[3]{x}}}_{\geq 0 \text{ on } [1,8],\ \text{so}}$
from $x = 1$ to $x = 8$.

$$\text{Area} = \int_1^8 \underbrace{\left(2 + \frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{Rewrite} \\ \text{Cont. on } [1,8] \checkmark}} dx$$

$$= \int_1^8 \underbrace{\left(2 + x^{-1/3}\right) dx}_{\text{Find an AD (1)}}$$

$$= \Big[\underbrace{2x + \frac{x^{2/3}}{2/3}}_{\text{Rewrite}}\Big]_1^8$$

$$= \left[2x + \frac{3}{2}x^{2/3}\right]_1^8$$

what's easier to do 1st? ² or ∛

$$= \left[2x + \frac{3}{2}\left(\sqrt[3]{x}\right)^2\right]_1^8$$

$$= \underbrace{\left[2(8) + \frac{3}{2}\left(\sqrt[3]{8}\right)^2\right]}_{\text{Eval. at 8}}$$

$$- \underbrace{\left[2(1) + \frac{3}{2}\left(\sqrt[3]{1}\right)^2\right]}_{\text{Eval. at 1}}$$

$$= \left[16 + \frac{3}{2}(2)^2\right] - \left[2 + \frac{3}{2}(1)^2\right]$$

$$= \left[16 + \frac{3}{\cancel{2}}\left(\cancel{4}^{2}\right)\right] - \left[2 + \frac{3}{2}\right]$$

$$= 22 - \frac{7}{2}$$

$$= \frac{44}{2} - \frac{7}{2}$$

2nd: Up to 69
3rd: Up to 75

$$= \boxed{\frac{37}{2} \text{ or } 18\frac{1}{2} \text{ square units}}$$

① Word Probs.

Idea $\int_a^b f'(t)\,dt = \underbrace{f(b) - f(a)}_{\text{“}[f(t)]_a^b\text{”}}$

$$\boxed{\begin{aligned}\int_a^b (\text{rate})\,dt &= \begin{pmatrix}\text{amt. at}\\ t=b\end{pmatrix} - \begin{pmatrix}\text{amt. at}\\ t=a\end{pmatrix}\\ &= \text{total (net) change from } a \text{ to } b\\ &\quad\text{“total accumulation”}\end{aligned}}$$

Book

Ex (#80, #86) — 2nd Ed., 3rd Ed.

(Shortened)

A car will sell at the rate of $\frac{30}{t}$ cars per month, where t is in months, and $t=1$ corresp. to Jan. 1. Find the # of cars that will be sold from Jan. 1 to May 1.

$t=1$ $t=5$

No IC here
Not like we're selling a ton of cars on Jan. 1

$$\int_1^5 \underbrace{\frac{30}{t}}_{\text{cont. on } [1,5] ✓}\,dt$$

$$= \int_1^5 30 \cdot \frac{1}{t}\,dt$$

$$= \Big[30 \ln \underbrace{|t|}_{\substack{\text{can drop}\\ t>0 \text{ on } [1,5]}}\Big]_1^5$$

$$= 30 \ln 5 - 30 \underbrace{\ln 1}_{=0}$$

$$= 30 \ln 5$$

$$\boxed{\approx 48 \text{ cars}}$$

(5.4) MORE APPLICS OF DEFINITE $\int$s

(A) Average Value of f (f_{av})

Prelim Ex

Quiz scores ✓?
How would I avg. your scores if Excel broke down?

Average of 4, 5, and 9

$$= \frac{4+5+9}{3} \quad \begin{matrix} \leftarrow \text{Sum} \\ \leftarrow \text{Input Size} \\ (\#\text{ of }\#\text{s}) \end{matrix}$$

$$= \boxed{6}$$

Idea

$$\underbrace{4+5+9}_{18} = \underbrace{6+6+6}_{18}$$

level off
Communism

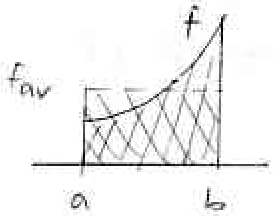
$18 to give to 3 kids

Like Price is Right range game - raise bar until get same total/area.

same area (18)

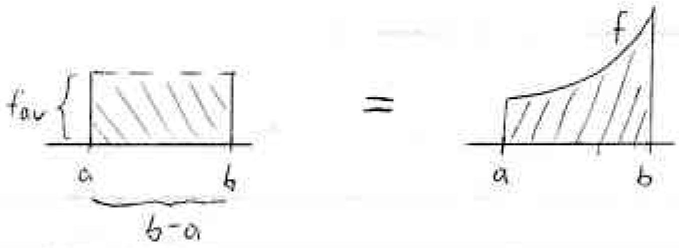
How do we find f_{av} for a func. f on $[a,b]$?
cont. on $[a,b]$

Ex



We need

We're merely extending the area principle from the discrete case.



How can we indicate the area using math notation?

$$(f_{av})(b-a) = \int_a^b f(x)\,dx$$

$$f_{av} = \frac{\int_a^b f(x)\,dx}{b-a}$$ ← Continuous Sum (Area) ← Input Size (Interval length)

Ex Find f_{av} of $f(x) = e^{2x}$ on $[-1, 2]$.

$$f_{av} = \frac{\int_{-1}^{2} e^{2x}\,dx}{2-(-1)}$$

$$= \frac{\left[\frac{e^{2x}}{2}\right]_{-1}^{2}}{3}$$ (Can pull out $\frac{1}{2}$)

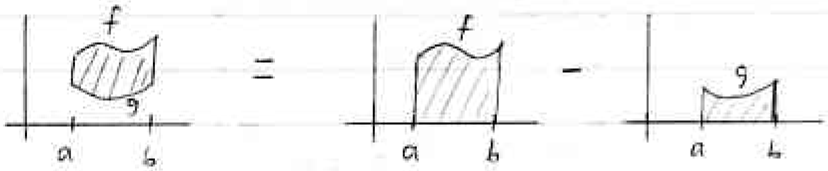
$$= \frac{\frac{e^{2(2)}}{2} - \frac{e^{2(-1)}}{2}}{3}$$

$$= \frac{\frac{1}{2}(e^{4} - e^{-2})}{3}$$

2nd, 3rd. Up to 29

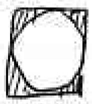
$$\boxed{\approx 9.077}$$

(B) Area Between Curves (No Intersections)



Like summing lengths of line segments
||||. $\int$ or cont. sum.
"Σxx" idea too intense? How can I describe the area using math...?

(Like Math 95:



Shaded area = Area of square − Area of circle

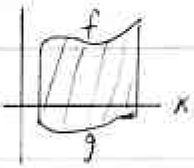
$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

↓ Linearity

$$= \int_a^b [f(x) - g(x)]\,dx$$

"top − bot."

Works even if

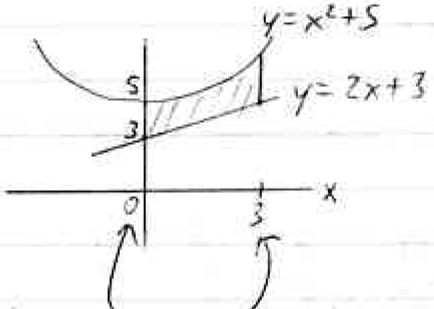


Ex $\left(\begin{matrix}2nd, 3rd \\ \#38\end{matrix}\right)$

They say "parabola" "line"
Book omits:

(a) Sketch $y = x^2 + 5$ and $y = 2x + 3$ on the same graph. (Hint: They never intersect.)

What's the point? $y = x^2 \to y = x^2 + 5$
Does the line go up or down?



(b) Find the area bet. them from $x = 0$ to $x = 3$.

$$\text{Area} = \int_0^3 [(x^2 + 5) - (2x + 3)]\,dx$$

top − bot.

$$= \int_0^3 [x^2 + 5 - 2x - 3]\,dx$$

$$= \int_0^3 [x^2 - 2x + 2]\,dx$$

$$= \left[\frac{x^3}{3} - x^2 + 2x\right]_0^3$$

$$= \left[\frac{(3)^3}{3} - (3)^2 + 2(3)\right] - [0]$$

2nd: Up to 37
3rd:

$$= \boxed{6 \text{ square units}}$$

ⓒ Area Between Curves (w/Intersections)

Ex (2nd, 3rd #44)

Find the area bounded by the curves
$y = 3x^2 - x - 1$ and $y = 5x + 8$.

We're setting up a def. ∫
What ?s should
we ask?

$$\int_a^b (\text{top} - \text{bot.})\,dx$$

What are a, b?

Which is which?

Intersection pts. corresp. to soln. of a ... system!

$y = y$

① To find a, b, find where the curves intersect

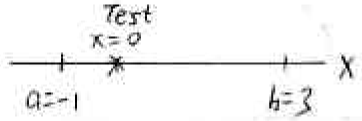
$$\text{Solve} \begin{cases} y = 3x^2 - x - 1 \\ y = 5x + 8 \end{cases}$$

$$3x^2 - x - 1 = 5x + 8$$
$$3x^2 - 6x - 9 = 0$$
$$3(x^2 - 2x - 3) = 0$$
$$3(x - 3)(x + 1) = 0$$
$$x = 3 \qquad x = -1$$

Who's a?

What's a convenient

② <u>Who's on top?</u>



(We have continuity for both.)
(Don't test 0 if 0 not in (a, b].)

$y = 3x^2 - x - 1$ (parabola) ← bottom
$= 3(0)^2 - (0) - 1$ at $x = 0$ (can skip)
$= -1$

8 > -1 or $y = 3x^2 - x - 1$ ↑ lead. coeff. > 0, so parabola opens up. ✓ Line intersects it twice. ⇒

Line must be on top of bounded region!

$y = 5x + 8$ (line) ← top
$= 5(0) + 8$ at $x = 0$
$= 8$

③ $\int$

$$\text{Area} = \int_{-1}^{3} \Big[\underbrace{(5x+8)}_{\text{top}} - \underbrace{(3x^2 - x - 1)}_{\text{bot.}} \Big] dx$$

$$= \int_{-1}^{3} [5x + 8 - 3x^2 + x + 1]\, dx$$

$$= \int_{-1}^{3} [-3x^2 + 6x + 9]\, dx$$

$$= \left[-x^3 + 6\left(\frac{x^2}{2}\right) + 9x \right]_{-1}^{3}$$

$$= \left[-x^3 + 3x^2 + 9x \right]_{-1}^{3}$$

$$= [-(3)^3 + 3(3)^2 + 9(3)] - [-(-1)^3 + 3(-1)^2 + 9(-1)]$$

$$= [27] - [-5]$$

$$= \boxed{32 \text{ square units}}$$

If census

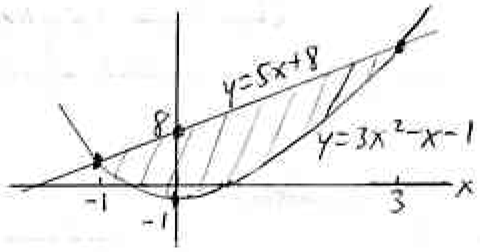
23

Use
$5x+8$
to compute

3

In fact, we found 4 pts. (x=0).
Like a developing photo
up to 45 (2nd, 3rd)

Turns out



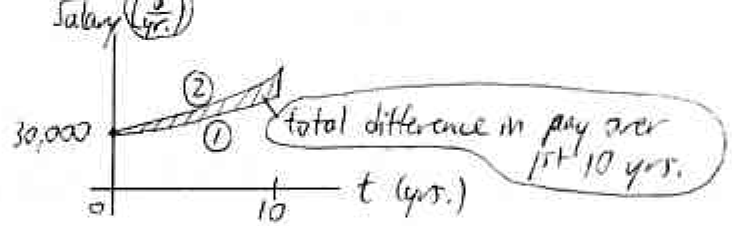
Ex (2nd, 3rd #56)
(Labor Contracts)

① Employer wants workers' salary (pay rate) to be $\$30{,}000e^{0.04t}$ per year.

② Union wants it to be $\$30{,}000e^{0.08t}$ per year.

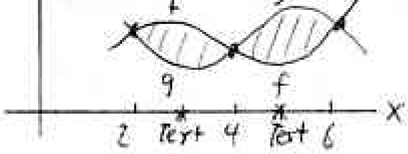
Book: $

Which is on top ① or ②
What employer wants or what union wants?



$$\int_0^{10} (30{,}000e^{0.08t} - 30{,}000e^{0.04t})\,dt \approx \$90{,}709.32$$

Harder



What what?

$$\text{Area} = \int_2^4 (f-g)\,dx + \int_4^6 (g-f)\,dx$$

5.6 : "u" SUBS

Mention Cover, Prod Quot. Make you so powerful $e^{\#}$, ln #

Chain Rule for D_x
$\approx$ u Subs for $\int$

<u>Idea</u> $\displaystyle\int \begin{pmatrix} \text{Ugly} \\ \text{in } x \end{pmatrix} dx = \int \begin{pmatrix} \text{Nice} \\ \text{in } u \end{pmatrix} du$

A Indefinite $\int$s

<u>Ex</u> $\displaystyle\int (x^3+1)^{10}\, x^2\, dx$

In PRINCIPLE Bin. Thm? w/ grandma

Mult. out? YUK! (Binomial Thm.)

Books always start "differential"

Let $u = x^3+1$

$\dfrac{du}{dx} = 3x^2$ (technically inappropriate to say we're mult. by dx)

$du = 3x^2\,dx$ "Differential form"

x^3 he doesn't want

We want to replace $3x^2dx$ with du. Put in 3

What makes up for the 3 Cancel with [=ETS] Make our own luck like dating

$$\int (x^3+1)^{10}\, x^2\, dx$$

$$= \underset{\substack{\text{to compensate} \\ \text{for 3}}}{\frac{1}{3}} \int (x^3+1)^{10} \cdot \underbrace{3x^2\,dx}_{du}$$

$$= \frac{1}{3}\int u^{10}\, du \qquad \text{(Nice!)}$$

What? + C (Put it?) Am I done? I give apples you give me worms

$$= \frac{1}{3} \cdot \frac{u^{11}}{11} + C$$

$$= \frac{1}{33} u^{11} + C$$

<u>Go back to x.</u>

If we had mult. out or Bin. Thm., we'd have to factor to get + C ⇒ some flexibility

$$= \boxed{\frac{1}{33}(x^3+1)^{11} + C} \qquad \text{Can } D_x \text{ to } \checkmark$$

You took me on faith here...

Strategies for Picking u

cooked up

$$\int \underbrace{(x^3+1)}_{u}{}^{10} x^2\, dx$$

non-0 (goes w/out saying)

① Its deriv. is (maybe) also in $\int$, except for a constant factor.

② u^{10} nice

It's buried

③ Common: u = "inside", exp., denom.

Templates: $\int u^n\, du$, $\int e^u\, du$, $\int \frac{du}{u}$

Another Way (not in book)

$x^2 dx$ is Bill

$$\int (x^3+1)^{10} \underbrace{x^2 dx}_{\text{"Kill"}}$$

$$u = x^3+1$$

$$du = 3x^2\, dx \Rightarrow \tfrac{1}{3}\, du = \underbrace{x^2 dx}_{\text{Replace "Kill" with}}$$

$$= \int u^{10} \cdot \tfrac{1}{3} du$$

$$= \tfrac{1}{3} \int u^{10}\, du$$

$\vdots$ (same as before)

Ex $\int \underbrace{(x^3+1)^{10}}_{u^{10}} \underbrace{x\ dx}_{\text{"Kill"}}$

~~Try $u = x^3+1$~~

~~$du = 3x^2\, dx$~~

(5)- Fancier tech

You can't put in an extra x

For a basic u-sub, "Kill" and du can differ by (at most) a <u>constant</u> factor.

7-11:

Can do 13-27 but we have more relevant Ex.

Ex $\int \sqrt{y-3}\ dy$

$$u = y-3$$
$$du = 1\,dy \text{ or } dy$$

$$= \int \sqrt{u}\ du$$
$$= \int u^{1/2}\ du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C$$
$$= \boxed{\frac{2}{3}(y-3)^{3/2} + C}$$

Ex $\int \frac{dx}{(4x-1)^3}$

$$u = 4x-1$$
$$du = 4\,dx$$

$$= \frac{1}{4}\int \frac{4\,dx}{(4x-1)^3}$$

$$= \frac{1}{4}\int \frac{du}{u^3}$$

$$= \frac{1}{4}\int u^{-3}\,du$$
$$= \frac{1}{4}\left(\frac{u^{-2}}{-2}\right) + C$$
$$= -\frac{1}{8}u^{-2} + C$$

3rd: Can do 13-27

$$= \boxed{-\frac{1}{8}(4x-1)^{-2} + C \quad \text{or} \quad -\frac{1}{8(4x-1)^2} + C}$$

Very cooked up!

Ex $\int (x^2+6x)^7 (x+3)\,dx$

$$u = x^2+6x$$
$$du = (2x+6)\,dx$$

I need to kill (x+3)dx How can I rewrite?

$$du = 2(x+3)\,dx$$

$$= \frac{1}{2}\int \underbrace{(x^2+6x)^7}_{u^7}\cdot \underbrace{2(x+3)\,dx}_{du}$$

$$= \frac{1}{2}\int u^7\,du$$
$$= \frac{1}{2}\cdot\frac{u^8}{8} + C$$
$$= \frac{1}{16}u^8 + C$$
$$= \boxed{\frac{1}{16}(x^2+6x)^8 + C}$$

3rd. Can do 13-45 but we have more relevant Exr.

8.5: 5 times the piece in front

Ex $\int 7x^3 e^{-2x^4}\,dx$

$$u = -2x^4$$
$$du = -8x^3\,dx$$

recip

$$= 7\left(-\frac{1}{8}\right)\int \underbrace{-8x^3 e^{-2x^4}\,dx}_{du}$$

$$= -\frac{7}{8}\int e^u\,du$$
$$= -\frac{7}{8}e^u + C$$

$$= \boxed{-\frac{7}{8}e^{-2x^4} + C}$$

Earlier ln prob later. ⓑ find indef ∫

Ex $\displaystyle\int \frac{x^2-3}{x^3-9x}\,dx$

$$u = x^3 - 9x$$
$$du = (3x^2-9)\,dx$$
$$du = 3(x^2-3)\,dx$$

$$= \frac{1}{3}\int \frac{\overbrace{3(x^2-3)\,dx}^{du}}{\underbrace{x^3-9x}_{u}}$$

$$= \frac{1}{3}\int \frac{du}{u}$$
$$= \frac{1}{3}\ln|u| + C$$
$$= \boxed{\frac{1}{3}\ln\left|\underbrace{x^3-9x}_{\text{Can be} < 0\text{, so don't drop } |\,|}\right| + C}$$

$-x - \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x|$

Long ÷, partial frac
M151, not 121
3rd: 4pt to 45

Ex $\displaystyle\int \overbrace{\frac{x^2}{1-x^2}\,dx}^{\text{"KILL"}}$

$$u = 1-x^2$$
$$du = -2x\,dx$$ (Need x^2, not x, to kill "KILL")

Can't do basic u-sub.

NO!: $\frac{1}{x}\int \cdots dx$

Never use something in as a "compensator" outside ∫.

≈ #37: Indef. ∫

(B) Definite ∫s

No PFs in 122?

I almost put 1 →

Bracing you for 122

Ex $$\int_2^3 \underbrace{\frac{x}{1-x^2}}_{\text{cont. on } [2,3] \checkmark} dx$$

$$u = 1 - x^2$$
$$du = -2x\,dx$$

New limits

$$x = 2 \Rightarrow u = 1-(2)^2 \Rightarrow u = -3$$
$$x = 3 \Rightarrow u = 1-(3)^2 \Rightarrow u = -8$$

Good reminder

$$= \left(-\frac{1}{2}\right)\int_{x=2}^{x=3} \frac{\overbrace{-2x\,dx}^{du}}{\underbrace{1-x^2}_{u}}$$

What's strange here? Don't technically need "u =" Areas same! $-x \Rightarrow -u$

$$= -\frac{1}{2}\int_{u=-3}^{u=-8} \frac{du}{u}$$

OK if lesser # is on top!

or $-\frac{1}{2}$ inside

$$= -\frac{1}{2}\Big[\ln|u|\Big]_{u=-3}^{u=-8}$$

Don't go back to x.

$$= -\frac{1}{2}\left(\ln|-8| - \ln|-3|\right)$$

or $\ln\sqrt{\frac{3}{8}}$

$$= \boxed{-\frac{1}{2}\left(\ln 8 - \ln 3\right) \text{ or } -\frac{1}{2}\ln\frac{8}{3}}$$

Improper ∫s in book: Change limits immed.

Another Way (helps in 122): Phase 1 Do Indefinite ∫ 1st: $\int \frac{x}{1-x^2}\,dx = \ldots = -\frac{1}{2}\int\frac{du}{u} = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|1-x^2| + C$ ($u = 1-x^2$) All sub work here!

You have to do the $1-x^2$ comps, anyway

Phase 2 $\Rightarrow \int_2^3 \frac{x}{1-x^2}\,dx \overset{\text{FTC}}{=} -\frac{1}{2}\Big[\ln|1-x^2|\Big]_2^3 = -\frac{1}{2}(\ln 8 - \ln 3)$

CH. 7: MULTIVARIABLE CALC

7.1: FUNCS. OF SEVERAL VARS.

How can this blow up?

Ex $f(x,y) = \frac{\sqrt{x-y}}{y}$

What do you call set of legal inputs?

(a) Find the [natural] domain of f.

We need: $x - y \geq 0$ and $y \neq 0$ to get real outputs.
$x \geq y$

$$\boxed{\{(x,y) \mid x \geq y,\ y \neq 0\}}$$

the set of all ... such that

Domain Issues
1. $\frac{1}{x}$ Need: $x \neq 0$
2. $\sqrt[\text{even}]{x}$ $x \geq 0$
3. $\ln x$ $x > 0$

Up to 5

(b) Find $f(7,3)$.

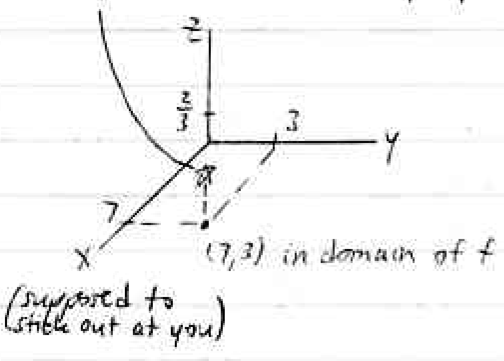
$f(7,3) = \frac{\sqrt{7-3}}{3}$ ← $\sqrt{4} = 2$

Up to 14 or 24

$= \boxed{\frac{2}{3}}$

Idea $(\overset{x}{7}, \overset{y}{3}, \overset{z}{\tfrac{2}{3}})$ lies on the graph of $z = f(x,y)$

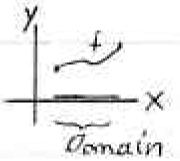
(1,1,1) ~ origin? NO (distortion)



Here, domain is 1D...

<u>Before</u> Graph $y=f(x)$

$$x \Rightarrow \boxed{f} \Rightarrow y$$

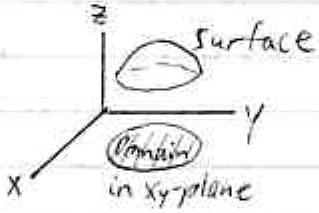


Calc would prompt you for 2 entries.

Why not sphere? Fails VLT.

<u>Now</u> Graph $z=f(x,y)$

$$(x,y) \Rightarrow \boxed{f} \Rightarrow z$$



Obsolete in 2 months?

<u>Ex</u> A store sells x Britney CDs at \$10 apiece,
y Ruben CDs at \$15 " , and
z Led Zep CDs at \$30 " .

What kind of func. can we find?

(a) Find the revenue func.

$$\boxed{R(x,y,z) = 10x + 15y + 30z}$$

Unless you're Steven Hawking

Can't see graph!

My bias

(b) If the store sells 3 Britney CDs, 10 Ruben CDs, and 100 Led Zep CDs, what's the revenue?

$$R(3,10,100) = 10(3) + 15(10) + 30(100)$$
$$= \boxed{\$3180}$$

(7.2): PARTIAL DERIVS.

(A) 1st-Order P.D.s

Before $y = f(x)$

Value of y depends on

y (dep. var.)

$|$ (f)

x (indep. var.)

Book allows $f = x^2$ but I don't like.

Ex y or $f(x) = x^2$

$\Rightarrow \frac{dy}{dx}$ or $f'(x)$

or $\frac{df}{dx}(x) = 2x$

Now $z = f(x,y)$

z (dep.)

/ (f) \

x (indep.) y (indep.)

Ex z or $f(x,y) = -5x^2 + y^3$

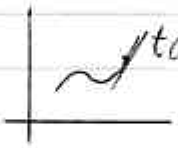
f_x or $\frac{\partial f}{\partial x}$ ("del") = the [partial] deriv. of f w/respect to x (treat y as a constant)

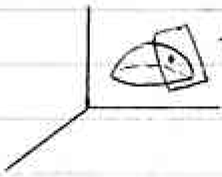
f_y or $\frac{\partial f}{\partial y}$ = " " y

(treat x as a constant)

Partials were slopes of what?

<u>Before</u> <u>Now</u>


tan line


tan plane (approxs. the graph near the pt.)

<u>Ex</u> $f(x,y) = -5x^2 + y^3$

$\Rightarrow f_x(x,y) = -10x$ (treat y^3 as a const.)

$f_y(x,y) = 3y^2$ (treat $-5x^2$ as a const.)

$f(1,2) = -5(1)^2 + (2)^3$
$= 3$
$\Rightarrow (1,2,3)$ lies on the graph of f

$f_x(1,2) = -10(1)$
$= -10$
$=$ slope of tan plane at $(1,2,3)$ in the x-direction

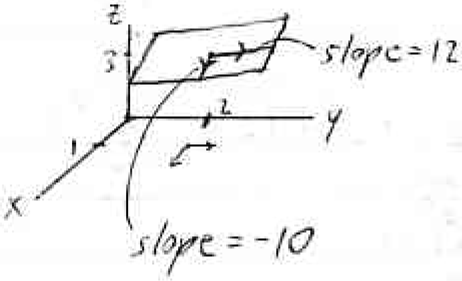
$f_y(1,2) = 3(2)^2$
$= 12$
$=$,
in the y-direction

We know now how the plane slants.

(Handout)

Eq. of plane: $z-3=-10(x-1)+12(y-2)$

Tan plane at $(1,2,3)$



(If $y \nearrow$ by 1, x fixed, then $z \nearrow$ by 12.)

Along this tan plane,
if $x \nearrow$ by 1 (and y stays fixed)
then f or $z \searrow$ by 10.

what a nice coincidence!

<u>Ex</u> The profit from producing x dolls and y flags is $P(x,y) = -5x^2 + y^3$ (in \$).

ⓐ Find the marginal profit func. for dolls.

$$P_x(x,y) = -10x$$

ⓑ Interpret $P_x(1,2)$.

$$P_x(1,2) = -10$$

"About" bec. talking about actual surface, not tan plane.

Profit decreases by about \$10 per addt'l doll (when 1 doll and 2 flags are produced).

Like 25

Ex $f(x,y) = xy^3 + \ln(2x - 3y^2)$

Find $f_x(x,y)$

$$f(x,y) = \underbrace{x}_{\substack{\text{"\#"} \\ \text{(treat as} \\ \text{const.)}}}\underbrace{y^3}\, + \ln(2x - \underbrace{3y^2}_{\text{"\#"}})$$

$D_x(6x)=6$
Do scratch work

$$f_x(x,y) = y^3 + \frac{1}{2x-3y^2}\underbrace{D_x(2x - \underbrace{3y^2}_{\#})}_{=2}$$

$$= \boxed{y^3 + \frac{2}{2x-3y^2}}$$

Find $f_y(x,y)$

$$f(x,y) = \underbrace{x}_{\#}y^3 + \ln(\underbrace{2x}_{\#} - 3y^2)$$

$$f_y(x,y) = x(3y^2) + \frac{1}{2x-3y^2}(-6y)$$

$$= \boxed{3xy^2 - \frac{6y}{2x-3y^2}}$$

Ex $f(x,y,z) = \underbrace{y}_{\#}e^{xy+yz}$. Find f_x and $f_x(0,3,4)$.

w
x y z

$$f_x(x,y,z) = ye^{xy+yz}\underbrace{D_x(x\underbrace{y}_{\#} + \underbrace{yz}_{\#})}_{y\,+\,0\,=\,y}$$

$$= \boxed{y^2 e^{xy+yz}}$$

$$f_x(0,3,4) = (3)^2 e^{(0)(3)+(3)(4)}$$
$$= \boxed{9e^{12}}$$

(B) 2^{nd}-Order P.D.s

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2}$$

$$\left.\begin{aligned} f_{xy} &= (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} \\ f_{yx} &= (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} \end{aligned}\right\} \text{usu.} =$$

$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

<u>Ex</u> $f(x,y) = (3x+y^2)^5$

should put (x,y)

$f_x = 5(3x+\underbrace{y^2}_{\#})^4(3)$ $f_y = 5(\underbrace{3x}_{\#}+y^2)^4(2y)$

D_x $(y=\#)$ $= \boxed{15(3x+y^2)^4}$ D_x $(y=\#)$ $= \boxed{10y(3x+y^2)^4}$

D_y $(x=\#)$

$f_{xx} = 60(3x+\underbrace{y^2}_{\#})^3(3)$ $f_{yx} = 10y \cdot 4(3x+\underbrace{y^2}_{\#})^3(3)$

What do you notice?
$f_{yx} = f_{xy}$

$= \boxed{180(3x+y^2)^3}$ $= \boxed{120y(3x+y^2)^3}$

=

$f_{xy} = 60(\underbrace{3x}_{\#}+y^2)^3(2y)$ f_{yy} (Product Rule)

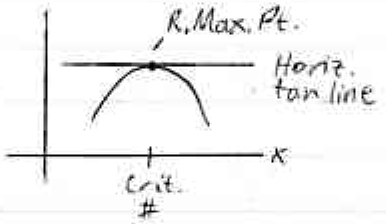
$= \boxed{120y(3x+y^2)^3}$

(7.3) : OPTIMIZATION

(A) Critical Points (CPs)

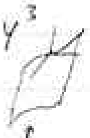
Before

What kind of tan line

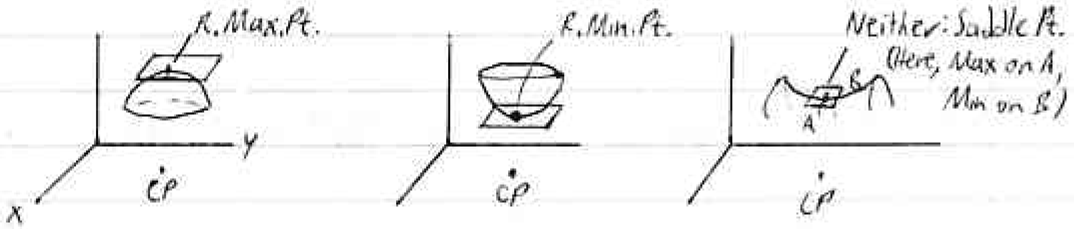


Now

$f(x,y) = y^3$



$f_x = 0$ still "saddle pts."



 Horiz. tan plane

(a,b) is a CP of $f(x,y)$ if
$f_x = 0$ and $f_y = 0$ there (or if either DNE).

guarantee a horiz. tan plane if f "nice"

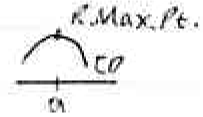
CPs are the only places where R. Max. and R. Min. Pts. can appear.

(B) Classifying CPs

f' never DNE

Before (2nd Deriv. Test for "nice" $f(x)$.)

If "a" is a crit. #,

If f'' or $f_{xx} < 0$ there $\Rightarrow$ 

$> 0 \Rightarrow$ 

$= 0 \Rightarrow$ test is useless

Larson 889: need cont. 2nd partials

Now (D-Test, or 2nd Deriv. Test for "nice" $f(x,y)$.)

If (a,b) is a CP,

Book doesn't do A, B, C

Let
$$A = f_{xx}$$
$$B = f_{xy}$$
$$C = f_{yy}$$
$$D = AC - B^2 \text{ at } (a,b)$$

Product of pure 2nd partials − product of 2nd mixed

Judge: "they're interesting"

(Case 1) If $D > 0$, then f has a R.Max. or a R.Min. at (a,b).

Larson 889 If $D>0 \Rightarrow$ A, C same sign $\Rightarrow$ can use A or C

(Case 1a) If $D > 0$ and $\underbrace{A < 0}_{f_{xx} < 0}$ $\frown$ CD

then f has a R.Max. at (a,b).

(Case 1b) If $D > 0$ and $\underbrace{A > 0}_{f_{xx} > 0}$ $\smile$ CU

then f has a R.Min. at (a,b)

Unlike old 2nd DT could you use " " to say "Neither?"

Case 2 If $D < 0$, then f has neither ("saddle pt.") at (a,b).

Judge shrugs.

Case 3 If $D = 0$, then the test is useless.

© Ex (#18)

Find the relative extreme values of
$$f(x,y) = -x^2 - y^3 - 6x + 3y + 4$$

Step 1: Find CPs

$$\left.\begin{aligned} f_x(x,y) &= -2x - 6 \\ f_y(x,y) &= -3y^2 + 3 \end{aligned}\right\} \underset{=0}{\text{set}}$$

Maybe system of linear eqs.

$$\text{Solve } \begin{cases} -2x - 6 = 0 \\ -3y^2 + 3 = 0 \end{cases}$$

$$\begin{aligned} -2x - 6 &= 0 \\ -2x &= 6 \\ x &= -3 \end{aligned} \quad \underline{\text{and}} \quad \begin{aligned} -3y^2 + 3 &= 0 \\ -3y^2 &= -3 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

CPs: $(-3, 1)$
$(-3, -1)$

Step 2: Find D

$f_x(x,y) = -2x-6$ $\qquad$ $f_y(x,y) = -3y^2+3$

$\Rightarrow A = f_{xx}(x,y) = -2$ $\qquad$ $\Rightarrow C = f_{yy}(x,y) = -6y$

$B = f_{xy}(x,y) = 0$

Judge func.

$$D = AC - B^2$$
$$= (-2)(-6y) - (0)^2$$
$$= 12y$$

Step 3: Classify CPs

$(-3,1)$

$D = 12y$ $\qquad$ $A = -2$
$= 12(1)$
$= 12$

$D > 0$ and $A < 0 \Rightarrow$ R. Max. at $(-3,1)$

$\cap$

$(-3,-1)$

$D = 12y$
$= 12(-1)$
$= -12$

$D < 0 \Rightarrow$ Neither at $(-3,-1)$

Step 4: Find extreme values

$f(-3,1) = -(-3)^2 - (1)^3 - 6(-3) + 3(1) + 4 = 15$

$\boxed{\text{R. Max. value of 15 at } (-3,1)}$

REVIEW 5.3-5.6

(5.3): Definite ∫s and Areas

FTC

who cares if not cont. at 0

Ex $\int_e^7 \frac{1}{x}\,dx$

cont. on [e, 7] ✓

May need algebra to rewrite.
Find an AD of $\frac{1}{x}$.
Don't need + C

$= [\ln|x|]_e^7$

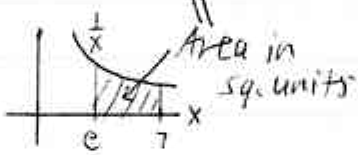
can drop
x>0 on [e, 7]

$= [\ln 7] - [\ln e]$

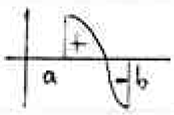
[Eval at top #] - [Eval at bot #]

May need

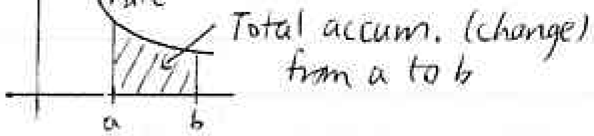
$= \boxed{\ln 7 - 1}$



Note

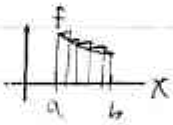


Word Probs.



Approx. $\int_a^b f(x)\,dx$ Using Left Riemann Sums

Given: $n = \#$ rects.



① Rect. width $\Delta x = \frac{b-a}{n}$

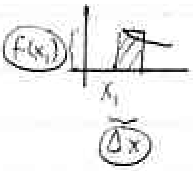
② Find

Keep $+\Delta x$ until you get n #s.

③ Left R. Sum

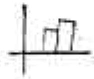
$$= (\text{Area of } 1^{st} \text{ rect.}) + (\quad 2^{nd} \quad) + \cdots + (\quad n^{th} \quad)$$

$$= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$



As $n \to \infty$, $\to$ Exact

Don't have to be same width - you can modify an approach

Note If given a table, can't use FTC, but can 

x	$f(x)$
1	20
2	30
⋮	⋮

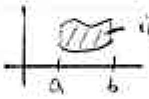
5.4 f_{av}

Sum Total / Input Size

$$= \frac{\int_a^b f(x)\,dx}{b-a}$$

$\frac{\text{Area}}{\text{Interval length}}$

Area Bet. Curves



in sq. units

May need

$$\int_a^b \left[(top)-(bot)\right]dx$$

Graph or test

to see who's who

I'll give it no intersecs.

find bounded area

To find a, b, solve $f(x)=g(x)$ for x.
Who's on top? Test

5.6 u-Subs

$\int f(x)\,dx$

Let u = inside, exp., denom.; its deriv. in $\int$ up to const. factor
$du = (\text{deriv.})\,dx$ (May need to factor)

Templates: $\int u^n\,du$, $\int e^u\,du$, $\int \frac{du}{u}$
Algebra?

Ex $\int \frac{x^3}{x^4+1}\,dx$

$u = x^4+1$
$du = 4x^3\,dx$

can put in const. factor

$= \frac{1}{4}\int \frac{4x^3}{x^4+1}\,dx$

Compensate

$= \frac{1}{4}\int \frac{du}{u}$
$= \frac{1}{4}\ln|u| + C$ (+C)
$= \frac{1}{4}\ln|x^4+1| + C$ (Go back to x.) ($\to x$)

can drop
$x^4+1>0$

Definite $\int$s: $\int_a^b f(x)\,dx$

$x = a \Rightarrow u =$
$\quad\; b$
No: (+C), ($\to x$)

or (Phase 1) Work out $\int f(x)\,dx$ 1st.
(Phase 2) $\int_a^b f(x)\,dx = [F(x)]_a^b$ (stick w/ x)
an AD from Phase 1

(Word Probs.)

REVIEW 7.1-7.3

7.1 $f(x, y, \ldots)$

Domain
Evaluate
Word probs.

7.2 $f_x, f_y, \ldots$
$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \ldots$

To find, treat other var. as #.
Evaluate

$f_{xx}, \underbrace{f_{xy}, f_{yx}}_{\text{usu.} =}, f_{yy}$ or $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$

Word probs.

7.3 Optimization

Finding CPs
Solve $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$

$D = AC - B^2$

$A \leftarrow f_{xx}$, $C \leftarrow f_{yy}$, $B \leftarrow f_{xy}$

Classify
If $D > 0$
If $A < 0$ $\cap$ R. Max. CD
If $A > 0$ $\cup$ R. Min. CU
If $D < 0 \Rightarrow$ neither
If $D = 0 \Rightarrow$ useless
Eval f to find extreme values.