CH. 7: MULTIVARIABLE CALC

7.1: FUNCS. OF SEVERAL VARS.

How can this blow up?

Ex $f(x,y) = \dfrac{\sqrt{x-y}}{y}$

What do you call set of legal inputs?

ⓐ Find the [natural] domain of f.

We need: $x - y \geq 0$ and $y \neq 0$ to get real outputs.
$x \geq y$

$$\boxed{\{(x,y) \mid x \geq y,\ y \neq 0\}}$$

the set of all ... such that

Domain Issues

①	$\frac{1}{x}$	Need: $x \neq 0$
②	$\sqrt[\text{even}]{x}$	$x \geq 0$
③	$\ln x$	$x > 0$

Up to 5

ⓑ Find $f(7,3)$.

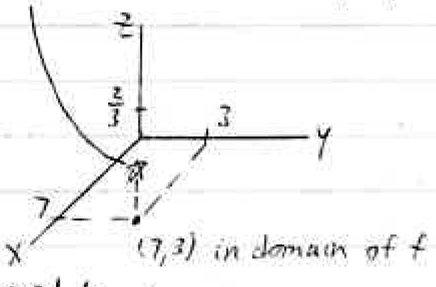
$f(7,3) = \dfrac{\sqrt{7-3}}{3}$ ← $\sqrt{4} = 2$

Up to 14 or 24

$= \boxed{\frac{2}{3}}$

Idea $(\overset{x}{7}, \overset{y}{3}, \overset{z}{\tfrac{2}{3}})$ lies on the graph of $z = f(x,y)$

(1,1,1) - origin? NO (distortion)



(supposed to stick out at you)

Here, domain is 1D...

Calc would prompt you for 2 entries.

Why not sphere? Fails VLT.

Obsolete in 2 months?

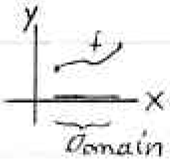
What kind of func. can we find?

Unless you're Steven Hawking

My bias

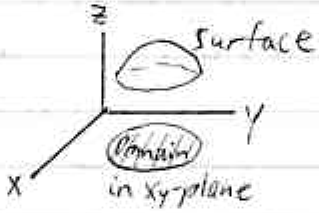
Before Graph $y=f(x)$

$$x \Rightarrow \boxed{f} \Rightarrow y$$



Now Graph $z=f(x,y)$

$$(x,y) \Rightarrow \boxed{f} \Rightarrow z$$



Ex A store sells x Britney CDs at \$10 apiece,
y Ruben CDs at \$15 " , and
z Led Zep CDs at \$30 " .

(a) Find the revenue func.

$$\boxed{R(x,y,z) = 10x + 15y + 30z}$$

Can't see graph!

(b) If the store sells 3 Britney CDs, 10 Ruben CDs, and 100 Led Zep CDs, what's the revenue?

$$R(3,10,100) = 10(3) + 15(10) + 30(100)$$
$$= \boxed{\$3180}$$

(7.2): PARTIAL DERIVS.

(A) 1st-Order P.D.s

Before $y = f(x)$

Value of y depends on

y (dep. var.)

$\mid (f)$

x (indep. var.)

Book allows $f = x^2$ but I don't like.

Ex y or $f(x) = x^2$

$\Rightarrow \frac{dy}{dx}$ or $f'(x)$
or $\frac{df}{dx}(x) = 2x$

Now $z = f(x, y)$

z (dep.)

$/ \; (f) \; \backslash$

x (indep.) $\quad y$ (indep.)

Ex z or $f(x,y) = -5x^2 + y^3$

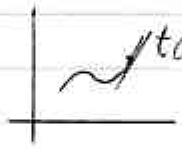
"del"

f_x or $\frac{\partial f}{\partial x}$ = the [partial] deriv. of f
w/respect to x
(treat y as a constant)

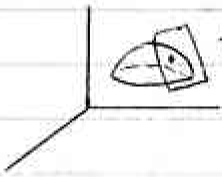
f_y or $\frac{\partial f}{\partial y}$ = "
y
(treat x as a constant)

Derivs were slopes of what?

<u>Before</u> <u>Now</u>



tan line



tan plane (approxs. the graph near the pt.)

<u>Ex</u> $f(x,y) = -5x^2 + y^3$

$\Rightarrow f_x(x,y) = -10x$ (treat y^3 as a const.)

$f_y(x,y) = 3y^2$ (treat $-5x^2$ as a const.)

$f(1,2) = -5(1)^2 + (2)^3$
$= 3$
$\Rightarrow (1,2,3)$ lies on the graph of f

$f_x(1,2) = -10(1)$
$= -10$
$=$ slope of tan plane at $(1,2,3)$ in the x-direction

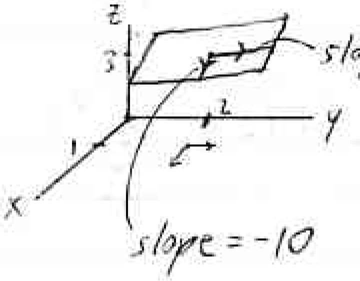
$f_y(1,2) = 3(2)^2$
$= 12$
$=$, in the y-direction

We know now how the plane slants.

(Handout)

Eq. of plane:
$z-3=-10(x-1)+12(y-2)$

Tan plane at $(1,2,3)$



(If $y\nearrow$ by 1, x fixed, then $z\nearrow$ by 12.)

Along this tan plane,
if $x\nearrow$ by 1 (and y stays fixed)
then f or $z\searrow$ by 10.

what a nice coincidence!

Ex The profit from producing x dolls and y flags is $P(x,y) = -5x^2+y^3$ (in \$).

(a) Find the marginal profit func. for dolls.

$$P_x(x,y) = -10x$$

(b) Interpret $P_x(1,2)$.

$$P_x(1,2) = -10$$

"About" bec. talking about actual surface, not tan plane.

Profit decreases by about \$10 per addt'l doll (when 1 doll and 2 flags are produced).

Like 25

<u>Ex</u> $f(x,y) = xy^3 + \ln(2x-3y^2)$

<u>Find $f_x(x,y)$</u>

$$f(x,y) = x\underbrace{y^3}_{\text{"\#" (treat as const.)}} + \ln(2x - \underbrace{3y^2}_{\text{"\#"}})$$

$D_x(6x)=6$
Do scratch work

$$f_x(x,y) = y^3 + \frac{1}{2x-3y^2} \underbrace{D_x(2x-3\underbrace{y^2}_{\#})}_{=2}$$

$$= \boxed{y^3 + \frac{2}{2x-3y^2}}$$

<u>Find $f_y(x,y)$</u>

$$f(x,y) = \underbrace{x}_{\#}y^3 + \ln(\underbrace{2x}_{\#}-3y^2)$$

$$f_y(x,y) = x(3y^2) + \frac{1}{2x-3y^2}(-6y)$$

$$= \boxed{3xy^2 - \frac{6y}{2x-3y^2}}$$

<u>Ex</u> $f(x,y,z) = \underbrace{y}_{\#}e^{xy+yz}$. Find f_x and $f_x(0,3,4)$.

w
x y z

$$f_x(x,y,z) = ye^{xy+yz}\underbrace{D_x(\underbrace{xy}_{\#}+\underbrace{yz}_{\#})}_{y+0=y}$$

$$= \boxed{y^2e^{xy+yz}}$$

$$f_x(0,3,4) = (3)^2e^{(0)(3)+(3)(4)}$$
$$= \boxed{9e^{12}}$$

(B) 2nd-Order P.D.s

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2}$$

$$\left.\begin{aligned} f_{xy} &= (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} \\ f_{yx} &= (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} \end{aligned}\right\} \text{usu.} =$$

$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

Ex $f(x,y) = (3x+y^2)^5$

should put (x,y)

$$f_x = 5(3x+\underbrace{y^2}_{\#})^4(3) \xrightarrow{D_x\,(y=\#)} = \boxed{15(3x+y^2)^4}$$

$$f_{xx} = 60(3x+\underbrace{y^2}_{\#})^3(3) = \boxed{180(3x+y^2)^3}$$

$$D_y\ (x=\#):\quad f_{xy} = 60(\underbrace{3x}_{\#}+y^2)^3(2y) = \boxed{120y(3x+y^2)^3}$$

$$f_y = 5(\underbrace{3x}_{\#}+y^2)^4(2y) \xrightarrow{D_x\,(y=\#)} = \boxed{10y(3x+y^2)^4}$$

$$f_{yx} = 10y\cdot 4(3x+\underbrace{y^2}_{\#})^3(3) = \boxed{120y(3x+y^2)^3}$$

f_{yy} (Product Rule)

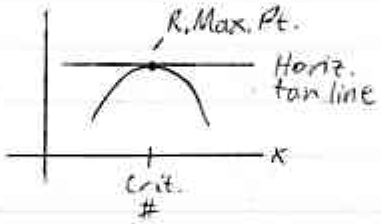
What do you notice? $f_{yx} = f_{xy}$

(7.3): OPTIMIZATION

(A) Critical Points (CPs)

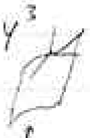
What kind of tan line

Before

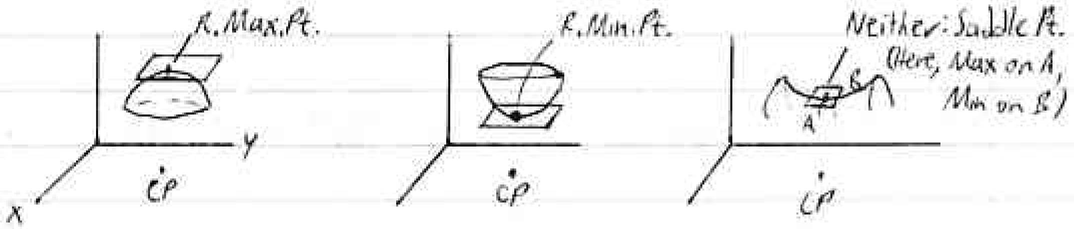


Now

$f(x,y) = y^3$



$f = 0$
still "saddle pts."



 Horiz. tan plane

(a,b) is a CP of $f(x,y)$ if
$f_x = 0$ and $f_y = 0$ there (or if either DNE).

guarantee a horiz. tan plane if f "nice"

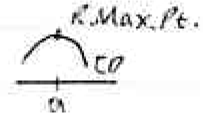
CPs are the only places where R. Max. and R. Min. Pts. can appear.

(B) Classifying CPs

f' never DNE

Before (2^{nd} Deriv. Test for "nice" $f(x)$.)

If "a" is a crit. #,

If f'' or $f_{xx} < 0$ there $\Rightarrow$ 

$> 0 \Rightarrow$ 

$= 0 \Rightarrow$ test is useless

Larson 889: need cont. 2nd partials

Now (D-Test, or 2^{nd} Deriv. Test for "nice" $f(x,y)$.)

If (a,b) is a CP,

Book doesn't do A, B, C

Let $A = f_{xx}$
$B = f_{xy}$
$C = f_{yy}$
$D = AC - B^2$ at (a,b)

Product of pure 2nd partials − product of 2nd mixed

D judge: "they're interesting"

(Case 1) If $D > 0$, then f has a R. Max. or a R. Min. at (a,b).

Larson 889 If $D > 0 \Rightarrow$ A, C same sign $\Rightarrow$ can use A or C

(Case 1a) If $D > 0$ and $\underbrace{A < 0}_{f_{xx} < 0}$ $\frown$ CD
then f has a R. Max. at (a,b).

(Case 1b) If $D > 0$ and $\underbrace{A > 0}_{f_{xx} > 0}$ $\smile$ CU
then f has a R. Min. at (a,b)

Unlike old 2nd DT could you use " " to say "Neither?"

(Case 2) If $D < 0$, then f has neither ("saddle pt.") at (a,b).

Judge shrugs.

(Case 3) If $D = 0$, then the test is useless.

© Ex (#18)

Find the relative extreme values of
$f(x,y) = -x^2 - y^3 - 6x + 3y + 4$

Step 1: Find CPs

$$\left.\begin{aligned} f_x(x,y) &= -2x - 6 \\ f_y(x,y) &= -3y^2 + 3 \end{aligned}\right\} \underset{=0}{\text{set}}$$

Maybe system of linear eqs.

$$\text{Solve } \begin{cases} -2x - 6 = 0 \\ -3y^2 + 3 = 0 \end{cases}$$

$$\begin{aligned} -2x - 6 &= 0 \\ -2x &= 6 \\ x &= -3 \end{aligned} \quad \underline{\text{and}} \quad \begin{aligned} -3y^2 + 3 &= 0 \\ -3y^2 &= -3 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

CPs: $(-3, 1)$
$(-3, -1)$

Step 2: Find D

$$f_x(x,y) = -2x-6 \qquad f_y(x,y) = -3y^2+3$$

$$\Rightarrow A = f_{xx}(x,y) = -2 \qquad \Rightarrow C = f_{yy}(x,y) = -6y$$

$$B = f_{xy}(x,y) = 0$$

Judge func.

$$\begin{aligned} D &= AC - B^2 \\ &= (-2)(-6y) - (0)^2 \\ &= 12y \end{aligned}$$

Step 3: Classify CPs

$(-3,1)$

$$\begin{aligned} D &= 12y \\ &= 12(1) \\ &= 12 \end{aligned} \qquad A = -2$$

$D > 0$ and $A < 0 \Rightarrow$ R. Max. at $(-3,1)$

$(-3,-1)$

$$\begin{aligned} D &= 12y \\ &= 12(-1) \\ &= -12 \end{aligned}$$

$D < 0 \Rightarrow$ Neither at $(-3,-1)$

Step 4: Find extreme values

$f(-3,1) = -(-3)^2 - (1)^3 - 6(-3) + 3(1) + 4 = 15$

$\boxed{\text{R. Max. value of 15 at } (-3,1)}$