SECTION 1.4: FUNCTIONS

(See p.40 for definitions of relations and functions and the Technical Note in Notes 1.24.)

Warning: The word "function" has different meanings in mathematics and in common English.

Unless otherwise specified, f, g, and h are assumed to be functions.

PART A: EXAMPLE

Consider a function f whose rule is given by $f(x) = x^2$.
As a short cut, we often say, "the function $f(x) = x^2$."

Warning: $f(x)$ is referred to as "f **of** x" or "f **at** x." It does **not** mean "f **times** x."

x is the input (or argument) for f, and x^2 is the output or function value.

$$x \rightarrow \boxed{f} \rightarrow x^2$$

This function squares its input, and the result is its output.

Note: The rule for this function could have been given as: $f(u) = u^2$, for example.

Example:

$$\begin{aligned} f(3) &= (3)^2 \\ &= 9 \end{aligned}$$

$$3 \rightarrow \boxed{f} \rightarrow 9$$

Example:

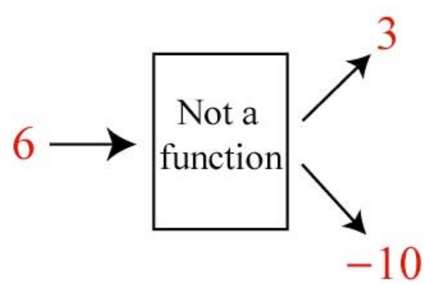
$$f(-3) = (-3)^2$$
$$= 9$$

Warning: It's often a good idea to use grouping symbols when you make a substitution (or "plug in"). Note that $f(-3)$ is **not** equal to -3^2, which equals -9. Remember that, in the absence of grouping symbols, exponentiation precedes multiplication (by -1 here) in the order of operations.

We can think of a function as a **calculator button**. In fact, your calculator should have a "squaring" button labeled x^2.

f is a function, because no "legal" input yields more than one output.

There is no function button on a calculator that ever outputs two or more values at the same time. The calculator never outputs, "I don't know. The answer could be 3 or -10."



Note: A function is a special type of relation. Relations that are not functions permit multiple outputs for a legal input.

PART B: SOME WAYS TO REPRESENT A FUNCTION

A function may be represented by …

1) an **algebraic statement** such as $f(x) = x^2$.

2) a **description in plain English** such as:

   "This function squares its input, and the result is its output."

3) an **input-output machine** such as:

$$x \rightarrow \boxed{f} \rightarrow x^2$$

4) a **table** of input x and output $f(x)$ values.

   Although it is often impossible to write a complete table for a function f, a partial table can be useful to look at, especially for graphing.

   If $f(x) = x^2$, we can write:

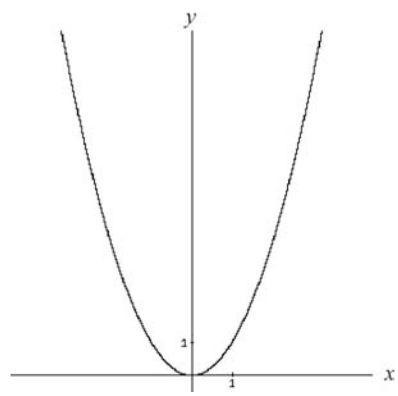
x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

5) a set of $\left(\text{input } x,\ \text{output } f(x)\right)$ **ordered pairs**.

For example, the table in 4) yields ordered pairs like $(-2, 4)$.

6) a **graph** of points that correspond to the ordered pairs in 5); see Section 1.5.

Here is a graph of $f(x) = x^2$:



7) an **arrow diagram** such as the one on p.40.

8) an **algorithm**.

Algorithms are discussed in Discrete Math: Math 245 at Mesa.

9) a **series**.

In Calculus: In Calculus II: Math 151 at Mesa, you will use series to represent functions. We will discuss series in Chapter 9. For example, the function $f(x) = \dfrac{1}{1-x}$ can be represented by the infinite series $1 + x + x^2 + x^3 + \ldots$, provided that $-1 < x < 1$. In Calculus, you will consider series for $\sin x$ and $\cos x$, among others.

etc.

PART C: DOMAIN AND RANGE

The domain of a function f, abbreviated $\text{Dom}(f)$, is the set of all "legal" **inputs**.

The range of f is then the set of all resulting **outputs**.

PART D: IMPLIED (OR NATURAL) DOMAIN

What is meant by "legal" inputs?

In geometry and in word problems, the application involved may require nonnegative and / or integer inputs. Even then, we might not worry about these restrictions when we apply calculus techniques. We sometimes use rounding.

Sometimes, we will choose a domain. We will discuss this in Section 1.9 on inverse functions.

Unless otherwise specified, we typically assume in Precalculus that the domain of a function is the set of **all real** input values that yield an output that is a **real** number. This set is the implied (or natural) domain.

The implied domain of an algebraic function, whose rule can be given by an algebraic expression (see Notes P.30), consists of all real numbers **except** those that:

1) lead to zero denominators $\left(\text{Think: } \frac{-}{0}\right)$, or

2) lead to negative radicands of even roots $\left(\sqrt[\text{even}]{-}\right)$.

Warning: This will be the case even after we review imaginary numbers.

As we study more types of functions, the list of restrictions will grow.
For example, as we will see later, we also exclude real numbers that lead to

3) logs of nonpositive values $\left(\log(\leq 0)\right)$, or

4) arguments of trig functions that correspond to vertical asymptotes.

PART E: INFINITY

The Harper Collins Dictionary of Mathematics defines infinity, denoted by ∞, as "a value **greater** than any computable value." The term "value" may be questionable!

Likewise, negative infinity, denoted by $-\infty$, is a "value" **lesser** than any computable value.

Warning: ∞ and $-\infty$ are **not** numbers. They are more conceptual. In higher math, we sometimes use the idea of a "point at infinity" in graphical settings.

PART F: EXAMPLES

Example 1

$$f(x) = x^2$$

The [implied] domain of **any** polynomial function (such as this f) is **R**, the set of all real numbers. In interval form, **R** is $(-\infty, \infty)$. Its graph is the entire real number line:

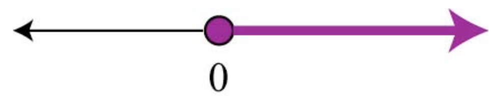
Warning: We use parentheses in the interval form, because ∞ ("infinity") and $-\infty$ ("negative infinity") are **not** real numbers and are therefore **excluded** from the set.

Warning: $f(x) = \dfrac{x^2 + x}{x}$, for example, is **not** a polynomial function, even though it simplifies to $f(x) = x + 1$ for **nonzero** values of x. The key is that its domain is not **R**.

The resulting range of f is the set of all nonnegative real numbers $\mathbf{R}_{\geq 0}$ (i.e., all real numbers that are greater than or equal to 0), because every such number is the square of some real number.

Warning: Squares of real numbers are never negative. This fact comes in very handy throughout math.

The graph of the range is:

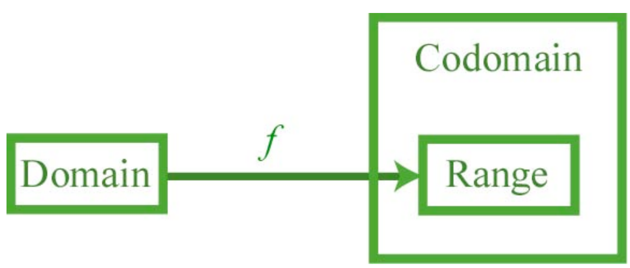


The filled-in circle serves to **include** 0 in the range. We could also use a left bracket ("[") here instead of a filled-in circle; the bracket opens towards the shading. The graph helps us figure out the interval form ….

In interval form, the range is $\left[0, \infty\right)$. We have a bracket next to the 0, because 0 is **included** in the range.

In set-builder form, the range is: $\left\{ y \,\middle|\, y \geq 0 \right\}$, or $\left\{ y : \ y \geq 0 \right\}$, which is read "the set of all [presumably real] values y such that $y \geq 0$." Using y instead of x is more consistent with our graphing conventions, and it helps us avoid confusion with the domain.

Technical Note: We say that f maps the domain **R** to the codomain **R**, or that f maps **R** to itself. Using notation, we write "$f : \mathbf{R} \rightarrow \mathbf{R}$." This is because f assigns a real number output (i.e., a member of the codomain) to **each** real number input in the domain. The range is a subset of the codomain. In fact, here, the range is a proper subset of the codomain, because not every real number in the codomain is assigned. In particular, the negative reals are not assigned.



Example 2

If $f(x)=\sqrt{x-3}$, find $\text{Dom}(f)$, the domain of f.

Solution

$f(x)$ yields real outputs $\Leftrightarrow$ $x-3\geq 0$.
The solution set of this inequality is the domain of f.

Solve the inequality:

$$x-3\geq 0$$
$$x\geq 3$$

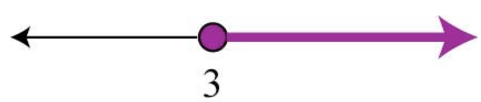
Remember that we may add or subtract the same real quantity to / from both sides of an inequality. Here, we added 3.

The domain of f …

… in set-builder form is:

$\{x \mid x\geq 3\}$, or $\{x : x\geq 3\}$

… in graphical form is:



… in interval form is:

$[3, \infty)$

Note: If the rule for f had been given by $f(t)=\sqrt{t-3}$, we're still talking about the same function. In particular, the domain and the range are still the same. However, when writing the domain in set-builder form, it may be more appropriate to write $\{t \mid t\geq 3\}$, or $\{t : t\geq 3\}$. The notation is self-contained, so we still could have used *x*, especially if *x* is not used elsewhere in the problem.

Example 3

If $f(x) = \sqrt{3 - x}$, find $\text{Dom}(f)$, the domain of f.

Solution

Solve the inequality: $3 - x \geq 0$.

Method 1

$$3 - x \geq 0$$

Subtract 3 from both sides.

$$-x \geq -3$$

Multiply or divide both sides by -1.

Warning: When we multiply or divide both sides of an inequality by the same negative real quantity, we must **reverse** the direction of the inequality symbol.

$$x \leq 3$$

Method 2

$$3 - x \geq 0$$

Add x to both sides.

$$3 \geq x$$

Switch the left side and the right side.

Warning: We must then **reverse** the direction of the inequality symbol.

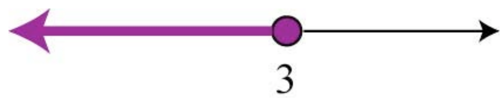
$$x \leq 3$$

The domain of f …

… in set-builder form is:

$\{x \mid x \leq 3\}$, or $\{x : x \leq 3\}$

… in graphical form is:



We could also use a right bracket ("]") here instead of a filled-in circle at 3.

… in interval form is:

$(-\infty, 3]$

Example 4

If $f(x) = \frac{1}{\sqrt{x-3}}$, find $\text{Dom}(f)$, the domain of f.

Solution

We must forbid a zero denominator here, so instead of solving the weak inequality $x - 3 \geq 0$, as we did in Example 2, we must solve the strict inequality: $x - 3 > 0$.

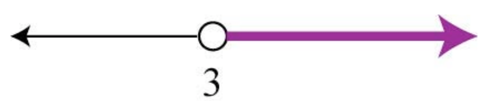
$$\begin{aligned} x - 3 &> 0 \\ x &> 3 \end{aligned}$$

The domain of f …

… in set-builder form is:

$\{x \mid x > 3\}$, or $\{x : x > 3\}$

… in graphical form is:



The hollow circle serves to **exclude** 3 from the domain. We could also use a left parenthesis ("(") here instead of a hollow circle at 3; the parenthesis opens towards the shading.

… in interval form is:

$(3, \infty)$

We have a parenthesis next to the 3, because 3 is **excluded** from the domain.

Types of Intervals

$(5, 7)$ and $(3, \infty)$ are examples of open intervals, because they **exclude** their endpoints. $(5, 7)$ is a bounded interval, because it is trapped between two numbers. $(3, \infty)$ is an unbounded interval.

$[5, 7]$ is an example of a closed interval, because it **includes** its endpoints, and it is bounded.

For more on intervals and inequalities, see Section A.1 (pp.A2-A3) and Section A.6 in Larson.

Example 5

If $f(x) = \sqrt[3]{x-3}$, find $\text{Dom}(f)$, the domain of f.

Solution

The domain of f is **R**, because:

- $x - 3$ is a polynomial, and

- (Warning!) The taking of **odd** roots (such as cube roots) does **not** impose any new restrictions on the domain. Remember that the cube root of a negative real number is a negative real number. This is different from **even** roots (such as square roots); we do not permit even roots of negative numbers when we find a domain.

For more on domains of radical functions, see Notes P.19-P.20.

Example 6

If $f(x)=\dfrac{\sqrt{x+3}}{x-10}$, find $\text{Dom}(f)$, the domain of f.

Solution

Because of the square root radical, we require:

$$\begin{aligned} x+3 &\geq 0 \\ x &\geq -3 \end{aligned}$$

Because we forbid zero denominators, we also require:

$$\begin{aligned} x-10 &\neq 0 \\ x &\neq 10 \end{aligned}$$

The domain of f …

… in set-builder form is:

$$\{x \mid x \geq -3 \text{ and } x \neq 10\}, \text{ or } \{x : x \geq -3 \text{ and } x \neq 10\}$$

… in graphical form is:

We include -3 but exclude 10.

… in interval form is:

$$[-3, 10) \cup (10, \infty)$$

The union symbol, $\cup$, is used to separate intervals in the event that a number or numbers need to be skipped.

We will discuss domain issues further in Section 1.8. We will discuss domains and nonlinear inequalities in Section 2.7.

PART G: PIECEWISE-DEFINED FUNCTIONS

These functions apply different rules for different values of x.

Example

Let $f(x) = |x|$. Recall the piecewise definition of absolute value (see Section A.1 and Notes P.16):

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Example

Let f be defined by:

$$f(x) = \begin{cases} x^2, & -2 \leq x < 1 \\ x+1, & 1 \leq x \leq 2 \end{cases}$$

To evaluate $f(-1)$, we use the top rule, since $-2 \leq -1 < 1$.

$$\begin{aligned} f(-1) &= (-1)^2 \\ &= 1 \end{aligned}$$

To evaluate $f(1)$, we use the bottom rule, since $1 \leq 1 \leq 2$.

$$\begin{aligned} f(1) &= (1) + 1 \\ &= 2 \end{aligned}$$

$f(10)$ is undefined, because we have no rule for $x = 10$.
10 is **not** in the domain of f, which is $[-2, 2]$.

In Calculus: You often see this type of function when you learn about limits and continuity in Calculus I (Chapter 2 in the Math 150 textbook at Mesa).

PART H: THE GREATEST INTEGER (OR FLOOR) FUNCTION

For the greatest integer function, $f(x) = [\![x]\!]$, where $[\![x]\!]$ is the largest integer that does not exceed x. Think of this as the "Price is Right" function: $[\![x]\!]$ is the closest integer that does not go over x.

This is also known as the floor function, $f(x) = \lfloor x \rfloor$. We basically round x **down**. If x is positive, we simply take the integer part.

Note: The ceiling function, $f(x) = \lceil x \rceil$, rounds x **up**.

Examples

$$[\![2.9]\!] = 2 \qquad \lfloor 2.9 \rfloor = 2$$

$$[\![3]\!] = 3 \qquad \lfloor 3 \rfloor = 3$$

$$[\![-1.9]\!] = -2 \qquad \lfloor -1.9 \rfloor = -2$$

Warning: Remember that noninteger inputs are usually legal! Don't just automatically assume that everything has to be an integer.

We may express this as a piecewise-defined function (in particular, a piecewise constant function) as follows:

$$f(x) = [\![x]\!] \text{ or } \lfloor x \rfloor = \begin{cases} \vdots & \\ -2, & -2 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \\ \vdots & \end{cases}$$

See p.69 for a graph of f. The function is a step function.

In Calculus: You may see this function when you learn about limits and continuity in Calculus I (Chapter 2 in the Math 150 textbook at Mesa).

PART I: DIFFERENCE QUOTIENTS

These typically have the form:

$$\frac{f(x+h)-f(x)}{h}, \text{ where } h \neq 0, \text{ or}$$

$$\frac{f(x)-f(a)}{x-a}, \text{ where } x \neq a$$

In both cases, we are taking the difference of outputs over the difference of inputs.

Example (First Form)

If $f(x)=\frac{1}{x}-3$, find $\frac{f(x+h)-f(x)}{h}$, where $h \neq 0$.

Solution

Warning: Here, as in many other cases, $f(x+h)$ is **not** equivalent to $f(x)+f(h)$.

Since $f(x)=\frac{1}{x}-3$, we have that: $f(x+h)=\frac{1}{x+h}-3$.
There are different ways of seeing this:

- Everywhere we see *x* in $f(x)=\frac{1}{x}-3$, we replace it with $x+h$, possibly using grouping symbols such as parentheses.

Warning: Such grouping symbols may be necessary.
For example, if we had $f(x)=x^2$, then $f(x+h)=(x+h)^2$.

• If you find it confusing to replace x with another expression in x, you may want to restate the function rule and perform a substitution.

For example, you could restate the rule here as $f(u) = \frac{1}{u} - 3$ and then use the substitution $u = x + h$. We then have: $f(x+h) = \frac{1}{x+h} - 3$.

In fact, this approach is closer to the idea of composite functions, which we will see in Section 1.8 and in Calculus.

• f takes its input, then takes its reciprocal, and then subtracts 3.

$$\frac{f(x+h) - f(x)}{h} = \frac{\left(\frac{1}{x+h} - 3\right) - \left(\frac{1}{x} - 3\right)}{h}$$

Warning: Don't forget the second set of parentheses in the N above. Otherwise, we will miss the "+" in the N below.

$$= \frac{\frac{1}{x+h} \cancel{-3} - \frac{1}{x} \cancel{+3}}{h}$$

$$= \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

Now, multiply the N and the D by the LCD, $x(x+h)$.

We would rather have a simple fraction than a complex (compound) fraction.

$$= \frac{x(x+h)\left(\frac{1}{x+h} - \frac{1}{x}\right)}{x(x+h)(h)}$$

Warning: Do **not** strike the expressions above in red until the Distributive Property is applied first! Then, we may do some striking (dividing out). If it helps, write out the first step below; with more experience, you may be able to skip it.

$$= \frac{x\cancel{(x+h)}\left(\frac{1}{\cancel{x+h}}\right) - \cancel{x}(x+h)\left(\frac{1}{\cancel{x}}\right)}{x(x+h)(h)}$$

$$= \frac{x-(x+h)}{xh(x+h)}$$

Warning: In the above fraction, remember the parentheses in the N. Otherwise, we will miss the "$-$" in the N below.

$$= \frac{\cancel{x} \cancel{-x} - h}{xh(x+h)}$$

$$= \frac{\cancel{-h}^{\,-1}}{x\cancel{h}(x+h)}$$

$$= -\frac{1}{x(x+h)}, \text{ where } h \neq 0$$

In the last step, we "canceled" (divided out) a pair of h factors.
Factoring and canceling (dividing) are key tricks for this kind of problem.

The restriction $h \neq 0$ is not evident from the final expression, $-\frac{1}{x(x+h)}$, so we state $h \neq 0$ explicitly.

Note: It is evident throughout that we require that $x \neq 0$. We also require that $x \neq -h$, but this kind of restriction is effectively avoided when you deal with difference quotients in Calculus.

Example (Second Form)

If $f(x)=\frac{1}{x}-3$, find $\frac{f(x)-f(2)}{x-2}$, where $x\neq 2$.

Solution

$$\frac{f(x)-f(2)}{x-2}=\frac{\left(\frac{1}{x}-3\right)-\left(\frac{1}{2}-3\right)}{x-2}$$

$$=\frac{\frac{1}{x}\cancel{-3}-\frac{1}{2}\cancel{+3}}{x-2}$$

$\left(\leftarrow \text{Easier than working out } \frac{1}{2}-3\right)$

$$=\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$$

Now, multiply the N and the D by the LCD, $2x$.

$$=\frac{2x\left(\frac{1}{x}-\frac{1}{2}\right)}{2x(x-2)}$$

$$=\frac{2\cancel{x}\left(\frac{1}{\cancel{x}}\right)-\cancel{2}x\left(\frac{1}{\cancel{2}}\right)}{2x(x-2)}$$

$$=\frac{\cancel{2-x}^{\,-1}}{2x\cancel{(x-2)}}$$

Because $2-x$ and $x-2$ are opposites and are nonzero (because we assume that $x\neq 2$), their quotient in either order is -1. See Notes P.37 on "Canceling" and the "Switch Rule for Subtraction."

$$=-\frac{1}{2x},\ \text{where } x\neq 2$$

We will discuss difference quotients further, as well as their graphical significance, in Section 1.5.

See Example 9 on p.46, which involves a polynomial, and Example 11 on p.A42 in Section A.4, which involves $f(x) = \sqrt{x}$.

Book Note: Leonhard Euler is profiled on p.42. He and Carl Gauss are considered by many to be the two greatest mathematicians of all time.