PART H: GRAPHING PIECEWISE-DEFINED FUNCTIONS

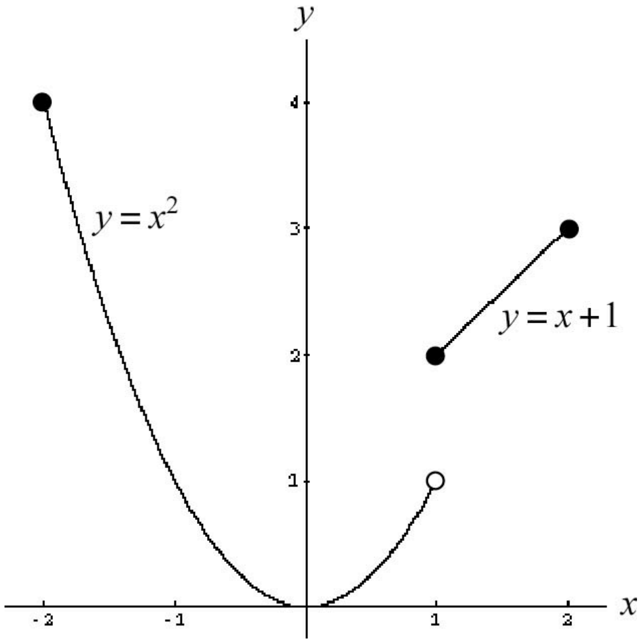
Example: See p.69 for the graph of $f(x) = [\![x]\!]$ or $\lfloor x \rfloor$. It is a piecewise constant function (in particular, a step function) with a piecewise horizontal graph.
See Section 1.4, Part H: Notes 1.32.

Example

Graph our function f from Section 1.4, Part G: Notes 1.31, where f is defined by:

$$f(x) = \begin{cases} x^2, & -2 \le x < 1 \\ x+1, & 1 \le x \le 2 \end{cases}$$

Its graph is below:



The top rule for $f(x)$ corresponds to the parabolic piece on the left.

The left endpoint at $x = -2$ is a **filled-in** circle, because it is **included** on the graph by way of the “weak” $\leq$ symbol.

The right endpoint at $x = 1$ is a **hollow** circle, because it is **excluded** from the graph by way of the “strict” $<$ symbol.

It helps to know that, at $x = 1$, x^2 **would have been** 1 in value, so that we place this circle at the point $(1, 1)$. In a way, for this purpose we are “borrowing” the top rule for the case $x = 1$, even though it technically does not apply.

The bottom rule corresponds to the line segment on the right.

Both endpoints are **filled-in** circles, because those points are **included** on the graph by way of the $\leq$ symbols.

<u>In Calculus</u>: When we study limits and continuity in Calculus (Chapter 2 in the Math 150 textbook at Mesa), we often study graphs with breaks like the one at $x = 1$ in the previous Example; that break is called a <u>jump discontinuity</u>.

<u>Technical Note</u>: We could **not** allow $-2 \leq x \leq 1$ to be the domain for the x^2 rule, given that $1 \leq x \leq 2$ is the domain for the $x + 1$ rule, because $f(1)$ would then not be well defined; it would be ambiguous. The VLT would have failed. In our Example, it is unambiguous that $f(1) = 2$, based on the bottom rule. However, remember that we did “borrow” the top rule for the purposes of locating the hollow circle at $x = 1$.

PART I: THE HORIZONTAL LINE TEST (HLT) (Also see Section 1.9.)

In Part B, Notes 1.40, we discussed the Vertical Line Test (VLT).

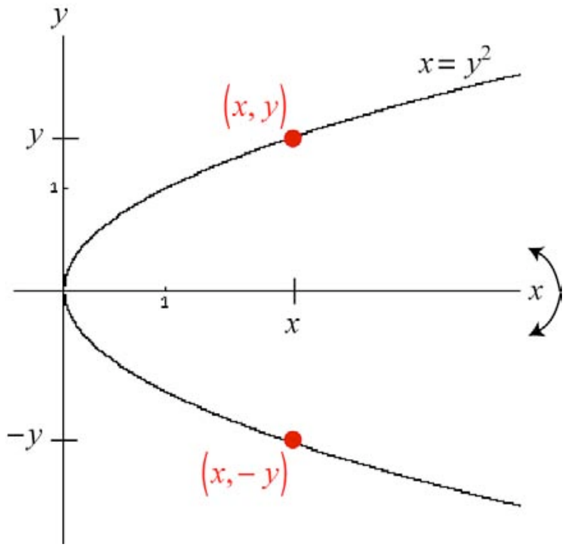
An equation in x and y describes $\boldsymbol{x}$ as a function of $\boldsymbol{y}$, and we can then say $\boldsymbol{x = f(y)}$ $\Leftrightarrow$
Its graph passes the **HLT** in the standard xy-plane, meaning that there is **no horizontal** line that intersects the graph more than once (i.e., there is no input $\boldsymbol{y}$ that corresponds to more than one output $\boldsymbol{x}$).

We have a shift in perspective if we allow $\boldsymbol{y}$ to be the **independent ("input")** variable and $\boldsymbol{x}$ to be the **dependent ("output")** variable.

In Calculus: You must be prepared for this shift in perspective, especially when you learn how to find areas, volumes, arc lengths, and surface areas in Chapter 6 of the Calculus I: Math 150 textbook at Mesa. The basic theory involved tends to stay the same, however.

Example

The graph of $x = y^2$ below passes the HLT, so it describes x as a function of y.



The graph is symmetric about the x-axis, because we get an equivalent equation if we replace y with $(-y)$:

$$x = (-y)^2$$

$$\Leftrightarrow \quad x = y^2$$

PART J: SYMMETRY AND EQUATIONS

See Section 1.2 of Larson: pp.18-19.

Consider an equation in x and y and its graph in the standard xy-plane.

1) The graph is symmetric about the y-axis ⇔
Replacing x with $\left(-x\right)$ yields an equivalent equation.

Example: $y = f\left(x\right)$, where f is an even function.
This is because the equations below are then equivalent:

$$\begin{aligned} & y = f\left(-x\right) \\ \Leftrightarrow \quad & y = f\left(x\right) \end{aligned}$$

2) The graph is symmetric about the x-axis ⇔
Replacing y with $\left(-y\right)$ yields an equivalent equation.

3) The graph is symmetric about the origin ⇔
Replacing x with $\left(-x\right)$ **and** y with $\left(-y\right)$ yields an equivalent equation.

Example: $y = f\left(x\right)$, where f is an odd function.
This is because the equations below are then equivalent:

$$\begin{aligned} & -y = f\left(-x\right) \\ \Leftrightarrow \quad & -y = -f\left(x\right) \qquad \left(\text{because } f \text{ is odd}\right) \\ \Leftrightarrow \quad & y = f\left(x\right) \end{aligned}$$

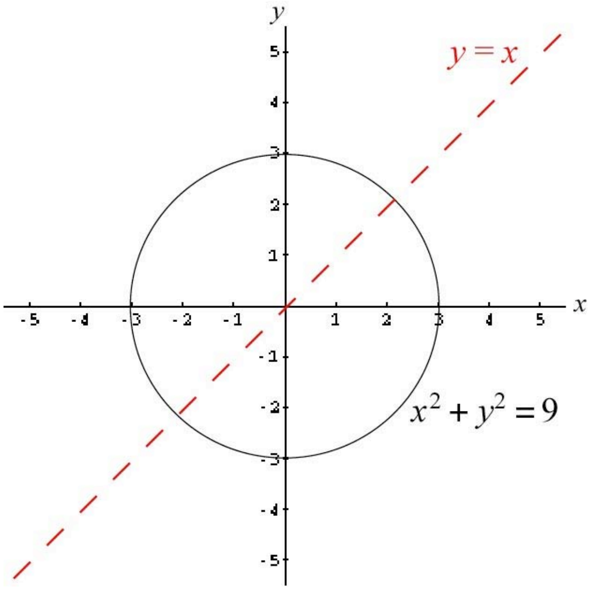
4) The graph is symmetric about the line $y = x$ ⇔
The equation is symmetric in x and y, meaning that
switching x and y throughout the equation yields an equivalent equation.

This will come up in Section 1.9, Part C.

In Calculus: 4) is a trick that can be used to simplify some procedures in Multivariable Calculus (Calculus III: Math 252 at Mesa) by trimming down repetitive manipulations.

Example

The graph of $x^2 + y^2 = 9$ exhibits all four aforementioned symmetries.



1) Symmetry about the *y*-axis

$$\begin{aligned} (-x)^2 + y^2 &= 9 \\ \Leftrightarrow \quad x^2 + y^2 &= 9 \end{aligned}$$

2) Symmetry about the *x*-axis

$$\begin{aligned} x^2 + (-y)^2 &= 9 \\ \Leftrightarrow \quad x^2 + y^2 &= 9 \end{aligned}$$

3) Symmetry about the origin

$$\begin{aligned} (-x)^2 + (-y)^2 &= 9 \\ \Leftrightarrow \quad x^2 + y^2 &= 9 \end{aligned}$$

4) Symmetry about the line $y = x$

$$\begin{aligned} y^2 + x^2 &= 9 \\ \Leftrightarrow \quad x^2 + y^2 &= 9 \end{aligned}$$

PART K: WHY DO WE CARE ABOUT DIFFERENCE QUOTIENTS? (OPTIONAL)

Remember Section 1.4, Part I: Notes 1.33-1.37. Reminder:

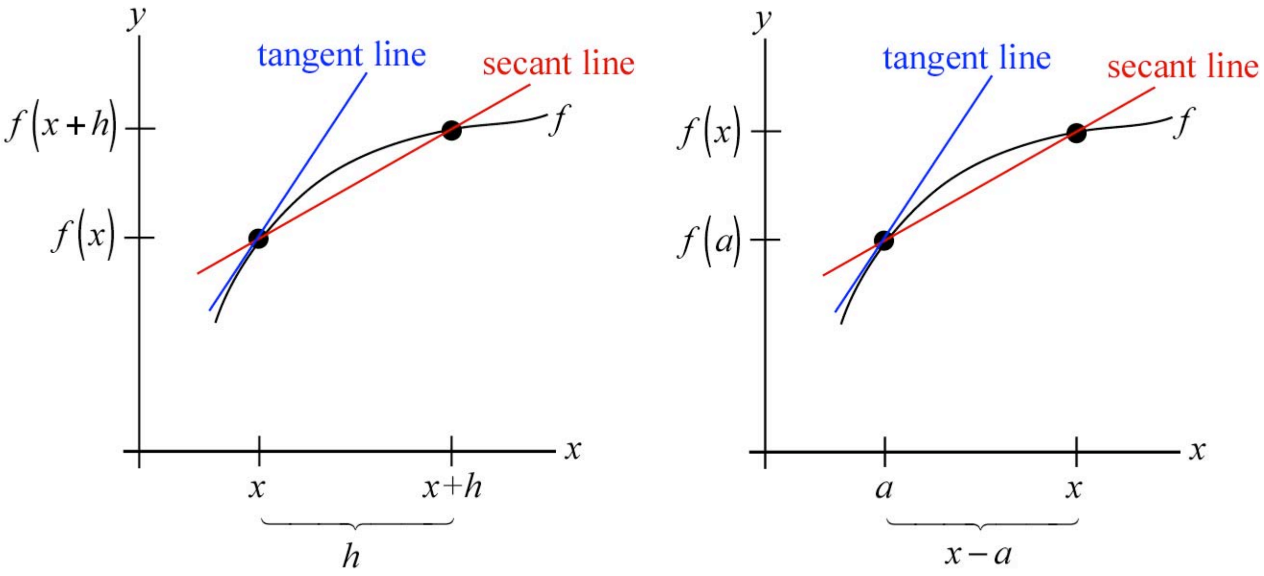
Difference quotients typically have the form:

$$\frac{f(x+h)-f(x)}{h}\text{, where } h \neq 0\text{, or}$$

$$\frac{f(x)-f(a)}{x-a}\text{, where } x \neq a$$

In both cases, we are taking the difference of outputs over the difference of inputs.

How are these two forms related to the two graphs below?



They represent the **slopes** of these red <u>secant lines</u> passing through two points on the graph of f . We will discuss these further in Part L.

We assume that the graph of f has no breaks or sharp corners "where we care."

In Calculus: You will take the limits of these expressions as h approaches 0 $(h \to 0)$ in the first form, and as x approaches a $(x \to a)$ in the second form. The result is the derivative, denoted by $f'(x)$ in the first form and $f'(a)$ in the second form, which is the slope of the tangent line to the point $(x, f(x))$ in the first graph and $(a, f(a))$ in the second graph. This is done in Calculus I (Chapter 3 of the Math 150 textbook at Mesa).

Example (In Calculus!)

In Section 1.4, Part I: Notes 1.33-1.35, we did the following:

If $f(x) = \frac{1}{x} - 3$, find $\frac{f(x+h) - f(x)}{h}$, where $h \neq 0$.

Our answer was: $-\frac{1}{x(x+h)}$, where $h \neq 0$.

What happens as h **approaches** 0? We will use limit notation.

$$\begin{aligned} \lim_{h \to 0} \left(-\frac{1}{x(x+h)} \right) &= -\frac{1}{x(x+0)} \\ &= -\frac{1}{x^2} \end{aligned}$$

Note: The restriction $x \neq 0$ is evident from this final expression.

We say that the derivative function (rule) $f'(x) = -\frac{1}{x^2}$. It is a “slope function” that gives you slopes of tangent lines to the graph of the original function at particular values of x. The above development suggests the following definition:

Definition 1: General Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ if it exists}$$

For example, let's find the slope of the tangent line to the graph of f at $x = 2$, that is, at the point $\left(2, -\frac{5}{2}\right)$; observe that $f(2) = -\frac{5}{2}$, so that is the *y*-coordinate of the corresponding point on the graph. The desired slope is the value of $f'(2)$:

Method 1

$$f'(x) = -\frac{1}{x^2}$$

$$\Rightarrow \quad f'(2) = -\frac{1}{(2)^2} = -\frac{1}{4}$$

Method 2

We may also find $f'(2)$ directly, without finding the general derivative $f'(x)$. We will use the following with $a = 2$.

Definition 2: Derivative at a Point

Let *a* be a real number in the domain of f.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \text{ if it exists}$$

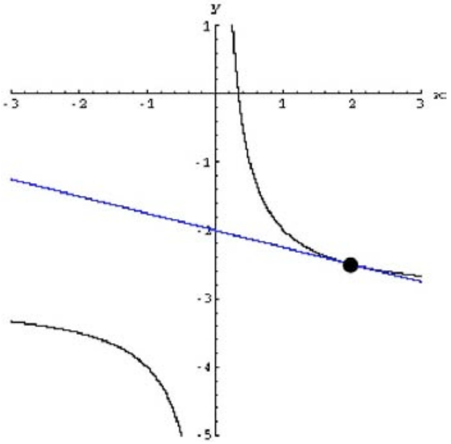
The omitted steps on the next page are found in Section 1.4: Notes 1.36. Do not worry about the restriction $x \neq 2$. When we take a limit as *x* **approaches** 2, we technically don't allow *x* to actually reach 2, itself.

$$
\begin{aligned}
f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \\
&\vdots \\
&= \lim_{x\to 2}\left(-\frac{1}{2x}\right) \\
&= -\frac{1}{2(2)} \\
&= -\frac{1}{4}
\end{aligned}
$$

Graph

The graph of $f(x)=\frac{1}{x}-3$ is in black below.

The tangent line of interest is in blue; it has slope $-\frac{1}{4}$.



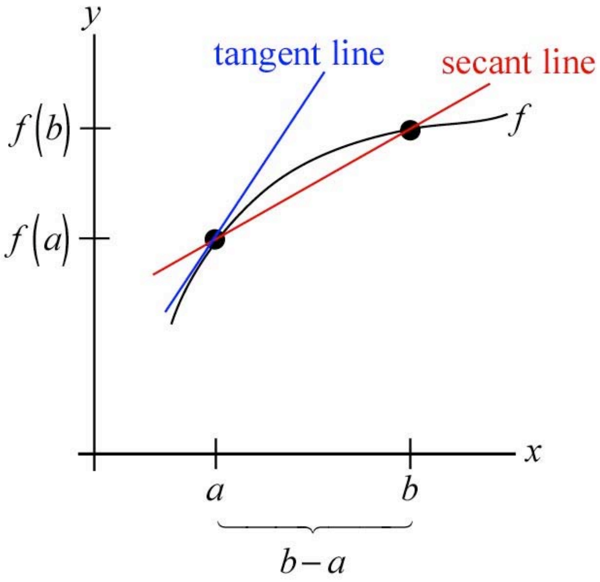
Warning: If the coordinate axes are not scaled the same, the slope will be visually distorted. Remember Notes 1.03. Equal scaling may be infeasible, but keep this in mind.

In Calculus: Derivatives are defined in Calculus I (Chapter 3 of the Math 150 textbook at Mesa), and partial derivatives for functions of more than one variable are defined in Calculus III (Math 252 at Mesa). These are **extremely** important in Calculus.

In Calculus: The "reverse" process of integration (defined in Chapter 5 of the Math 150 textbook at Mesa) involves finding functions that have a given function as their derivative.

PART L: DIFFERENCE QUOTIENTS AND RATES OF CHANGE (AVERAGE AND INSTANTANEOUS)

We will modify one of the graphs in Notes 1.63 by replacing *x* with *b*:



What is the significance of the slope of the red secant line above? It is the …

Average Rate of Change of f on the Interval $[a, b]$

This is given by: $\dfrac{f(b)-f(a)}{b-a}$.

As in Part K, we assume that the graph of f has no breaks or sharp corners on the interval.

Example

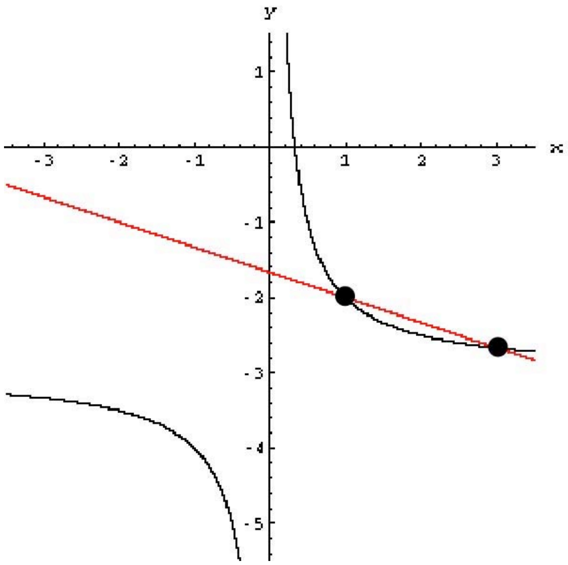
If $f(x) = \frac{1}{x} - 3$, find the average rate of change of f on the interval $[1, 3]$.

Solution

$$\frac{f(3) - f(1)}{3 - 1} = \frac{\left(\frac{1}{3} - 3\right) - \left(\frac{1}{1} - 3\right)}{2}$$

$$= -\frac{1}{3}$$

This is the slope of the red secant line below:



A Comment on Units

If, say, x is measured in hours and y is measured in miles, then we could say that the answer is $-\frac{1}{3}$ miles per hour (mph). In general, when we graph $y = f(x)$, the units we attach to slopes are of the form $\frac{\text{units for } y}{\text{units for } x}$.

In fact, derivatives represent instantaneous rates of change.

Follow-Up Example (Optional)

If $f\left(x\right)=\frac{1}{x}-3$, the instantaneous rate of change at $x=2$ is $-\frac{1}{4}$, as we found in Part K. It is the slope of the blue tangent line below.