CHAPTER 2: POLYNOMIAL AND RATIONAL FUNCTIONS

SECTION 2.1: QUADRATIC FUNCTIONS (AND PARABOLAS)

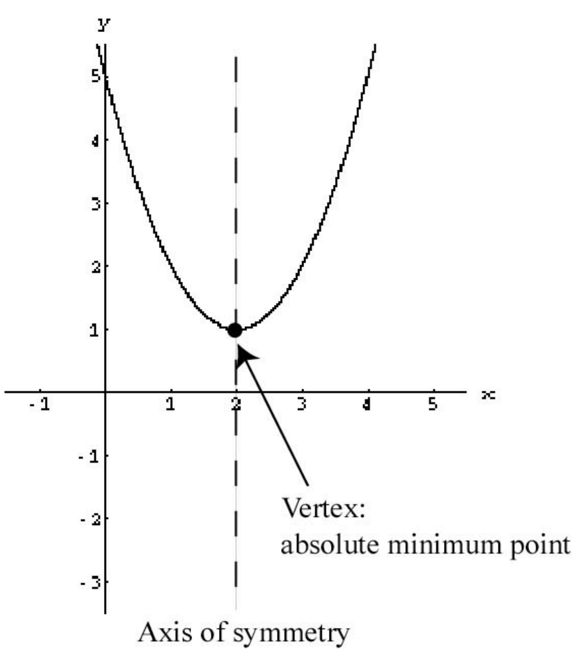
PART A: BASICS

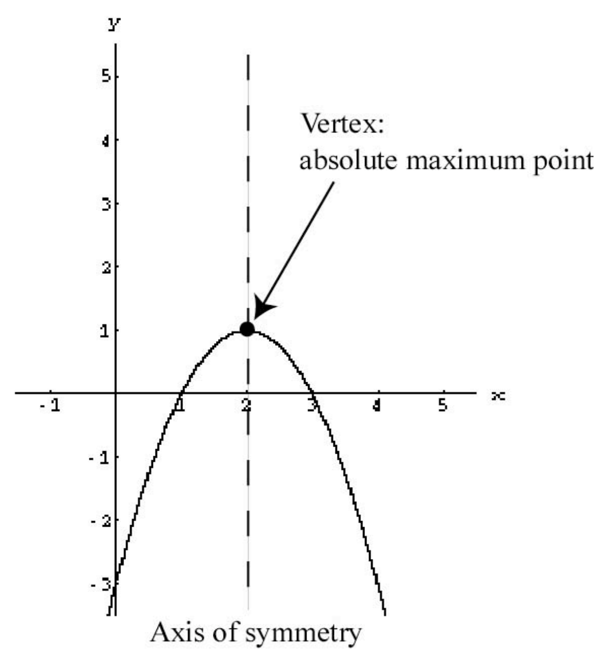
If a, b, and c are real numbers, then the graph of $\underbrace{f(x)}_{=y} = ax^2 + bx + c$ is a parabola, provided $a \neq 0$.

If $a > 0$, it opens **upward**.
If $a < 0$, it opens **downward**.

Examples

The graph of $y = x^2 - 4x + 5$ (with $a = 1 > 0$) is on the left.
The graph of $y = -x^2 + 4x - 3$ (with $a = -1 < 0$) is on the right.





PART B: FINDING THE VERTEX AND THE AXIS OF SYMMETRY (METHOD 1)

The vertex of the parabola [with equation] $y = ax^2 + bx + c$ is (h, k), where:

x-coordinate $= h = -\dfrac{b}{2a}$, and

y-coordinate $= k = f(h)$.

The axis of symmetry, which is the vertical line containing the vertex, has equation $x = h$.

(Does the formula for *h* look familiar? We will discuss this later.)

Example

Find the vertex of the parabola $y = x^2 - 6x + 5$. What is its axis of symmetry?

Solution

The vertex is (h, k), where:

$$h = -\frac{b}{2a} = -\frac{-6}{2(1)} = \mathbf{3}, \text{ and}$$

$$\begin{aligned} k &= f(3) \\ &= (3)^2 - 6(3) + 5 \\ &= \mathbf{-4} \end{aligned}$$

The vertex is $(\mathbf{3, -4})$.

The axis of symmetry has equation $\boldsymbol{x = 3}$.

Since $a = 1 > 0$, we know the parabola opens **upward**. Together with the vertex, we can do a basic sketch of the parabola.

PART C: FINDING MORE POINTS

"Same" Example: $y = x^2 - 6x + 5$, or $f(x) = x^2 - 6x + 5$

Find the y-intercept.

Plug in 0 for x. Solve for y.
In other words, find $f(0)$.
The y-intercept is **5**.

(If the parabola is given by $y = ax^2 + bx + c$, then c, the constant term, is the y-intercept. Remember that b was the y-intercept for the line given by $y = mx + b$.)

Find the x-intercept(s), if any.

Plug in 0 for y. Solve for x (only take **real** solutions).
In other words, find the **real** zeros of $f(x) = x^2 - 6x + 5$.

$$0 = x^2 - 6x + 5$$
$$0 = (x-5)(x-1)$$
$$x = 5 \quad \text{or} \quad x = 1$$

The x-intercepts are **1** and **5**.

Note: Observe that $f(x) = x^2 + 1$ has no real zeros and, therefore, has no x-intercepts on its graph.

You can find other points using the Point-Plotting Method (Notes 1.27).
Symmetry helps!

PART D: PARABOLAS AND SYMMETRY

"Same" Example

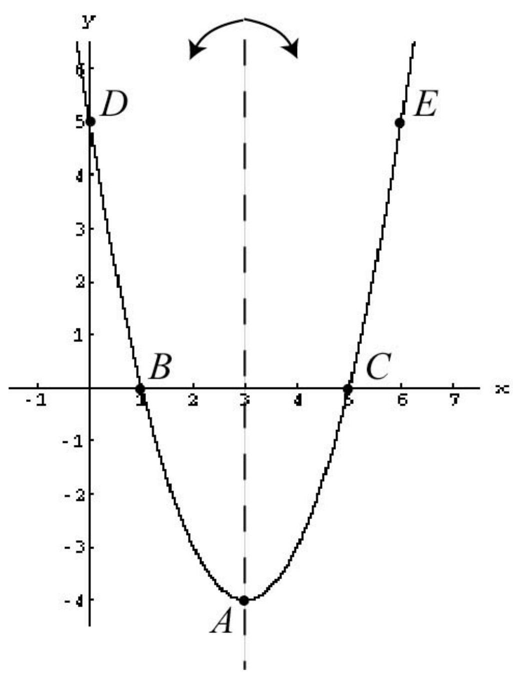
Sketch the graph of $y = x^2 - 6x + 5$, or $f(x) = x^2 - 6x + 5$.
Clearly indicate:

1) The vertex
2) Which way the parabola opens
3) Any intercepts

Make sure your parabola is symmetric about the axis of symmetry.

Solution

Point A is the vertex.
Points B and C are the x-intercepts.
Point D is the y-intercept.
We get the "bonus" Point E by exploiting symmetry.



Observe that the *x*-intercepts are symmetric about the axis of symmetry.
This makes sense, because the zeros of f are given by the QF:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The average of these zeros is $-\frac{b}{2a}$, which is the *x*-coordinate for the vertex and the axis of symmetry.

Technical Note: If the zeros of a quadratic $f(x)$ are imaginary, then they must have the same real part, which will be $-\frac{b}{2a}$. The axis of symmetry will still be $x = -\frac{b}{2a}$. There will be no *x*-intercepts, however.

Technical Note: The *x*-intercept on the right (if any) does not always correspond to the "+" case in the QF. Remember, *a* can be negative.

PART E: STANDARD FORM FOR THE EQUATION OF A PARABOLA; FINDING THE VERTEX AND THE AXIS OF SYMMETRY (METHOD 2)

The graph of $y = a(x-h)^2 + k$ is a parabola that has the same shape as the parabola $y = ax^2$, but its vertex is (h, k).

Remember translations from Section 1.6.
The parabola $y = ax^2$, which has vertex $(0, 0)$, is shifted *h* units horizontally and *k* units vertically to obtain the new parabola.

If $y = a(x-h)^2 + k$ is written out in the form $y = ax^2 + bx + c$, then the "*a*"s are, in fact, the same.

Example

Rewrite $y = 3x^2 + 12x + 11$ in the form $y = a(x-h)^2 + k$, and find the vertex.

Solution

Step 1: Group the quadratic and linear terms and factor out a from the group.

$$y = \left(3x^2 + 12x\right) + 11$$
$$y = 3\left(x^2 + 4x\right) + 11$$

Step 2: Complete the Square (CTS) **and Compensate**.

Take the coefficient of *x* inside the (), halve it, and square the result. This is the number we add inside the ().

Half of 4 is 2, and the square of 2 is 4.

$$y = 3\underbrace{\left(x^2 + 4x + 4\right)}_{\text{PST}} + 11 - 3(4)$$

Step 3: Factor our Perfect Square Trinomial (PST) as the square of a binomial, and simplify.

$$\mathbf{y = 3(x + 2)^2 - 1}$$

The vertex is $(\mathbf{-2, -1})$.

Warning: Don't forget the first "$-$." Remember that the *x*-coordinate of the vertex must make $(x + 2)$ equal to 0. This makes sense, because the vertex is an "extreme" point, and 0 is as "extreme" a square can get.

The axis of symmetry is $x = -2$.
$a = 3 > 0$, so the parabola opens upward.

Note: A key advantage that the form $y = a(x - h)^2 + k$ has over the form $y = ax^2 + bx + c$ is that the vertex is easier to find using the first form. Also, *x*-intercepts (but not the *y*-intercept) may be easier to find.

PART F: GIVEN SOME POINTS, FIND THE PARABOLA PASSING THROUGH THEM

Two distinct points determine a line.

Three noncollinear points (i.e., points that do not lie on a straight line) determine a parabola.

If we know the **vertex** and one other point on the parabola, then we can get a third point automatically by exploiting symmetry.

Technical Note: If you successively plug in the coordinates of the three points into the form $y = ax^2 + bx + c$, you will obtain a system of three linear equations in three unknowns (a, b, and c). Have fun!

If we know the vertex, then the form $y = a(x-h)^2 + k$ makes matters easier for us.

Example

Find an equation for the parabola that has vertex $(4, 2)$ and that contains the point $(1, 29)$.

Solution

Using the vertex:

$$y = a(x-h)^2 + k$$
$$y = a(x-4)^2 + 2$$

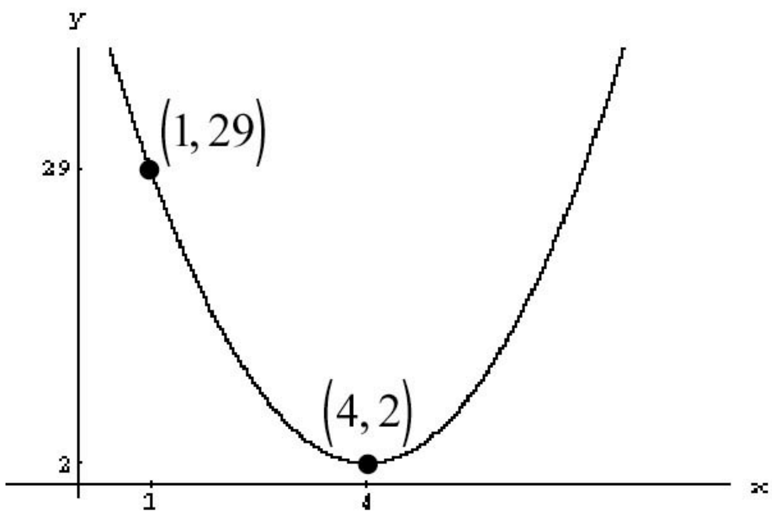
Plug in $(x = 1, y = 29)$ and solve for a:

$$29 = a(1-4)^2 + 2$$
$$29 = 9a + 2$$
$$27 = 9a$$
$$a = 3$$

An equation for the parabola is:

$$y = 3(x-4)^2 + 2$$

Here's a graph:



Note: In applications, you can think of this as "modeling a quadratic function."

We will see parabolas again in Section 10.2.