SECTION 2.2: POLYNOMIAL FUNCTIONS OF HIGHER DEGREE

PART A: INFINITY

The Harper Collins Dictionary of Mathematics defines infinity, denoted by ∞, as "a value **greater** than any computable value." The term "value" may be questionable!

Likewise, negative infinity, denoted by $-\infty$, is a value **lesser** than any computable value.

Warning: ∞ and $-\infty$ are **not** numbers. They are more conceptual. We sometimes use the idea of a "point at infinity" in graphical settings.

PART B: LIMITS

The concept of a limit is arguably the key foundation of calculus.
(It is the key topic of Chapter 2 in the Calculus I: Math 150 textbook at Mesa.)

Example

"$\lim_{x \to \infty} f(x) = -\infty$" is read "the limit of $f(x)$ as x approaches infinity is negative infinity." It can be rewritten as:

"$f(x) \to -\infty$ as $x \to \infty$," which is read "$f(x)$ approaches negative infinity as x approaches infinity."

The Examples in Part C will help us understand these ideas!

PART C: BOWLS AND SNAKES

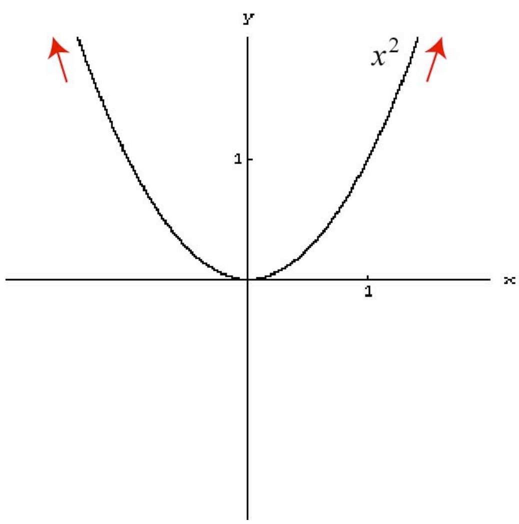
Let a represent a nonzero real number.

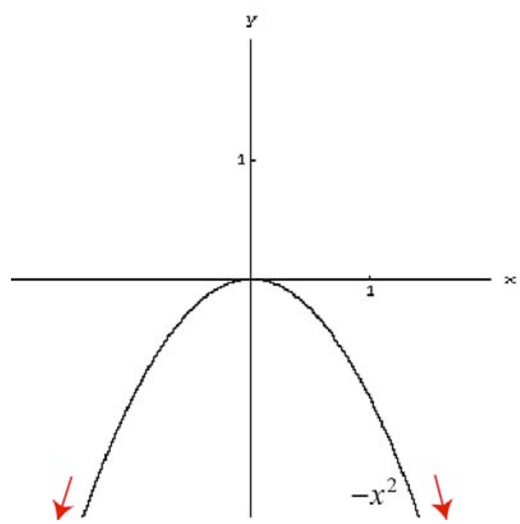
Recall that the graphs for ax^2, ax^4, ax^6, ax^8, etc. are "bowls."

If $a > 0$, then the bowls open **upward**.
If $a < 0$, then the bowls open **downward**.

Examples

The graph of $f(x) = x^2$ is on the left, and the graph of $g(x) = -x^2$ is on the right.





$$\lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to\infty} g(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to-\infty} g(x) = -\infty$$

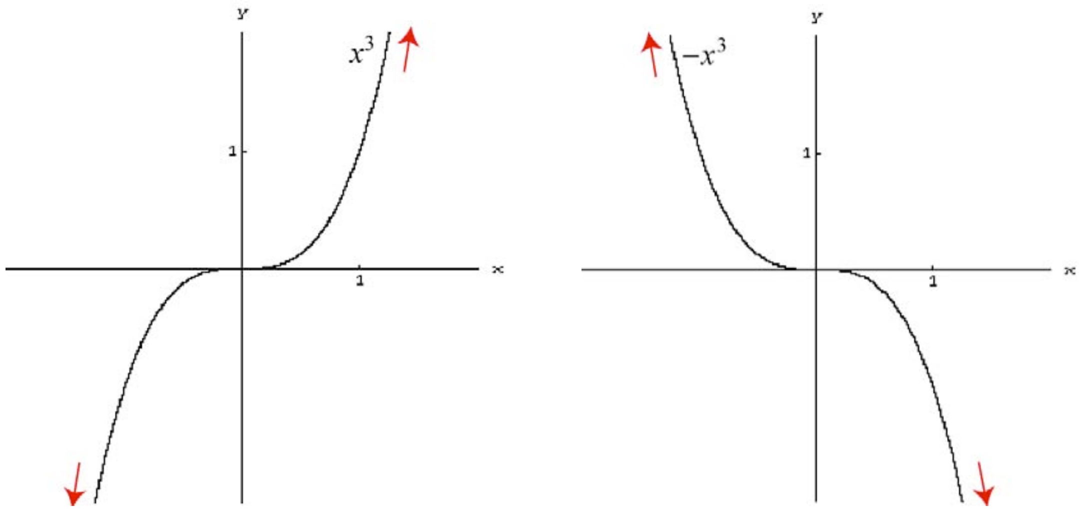
Let a represent a nonzero real number.

Recall that the graphs for ax^3, ax^5, ax^7, ax^9, etc. are "snakes."

If $a > 0$, then the snakes **rise** from left to right.
If $a < 0$, then the snakes **fall** from left to right.

Examples

The graph of $f(x) = x^3$ is on the left, and the graph of $g(x) = -x^3$ is on the right.



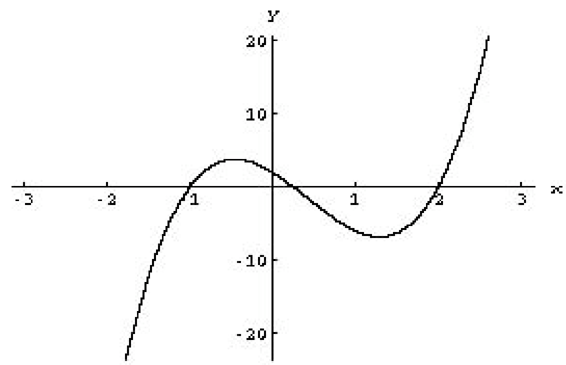
$\lim_{x\to\infty} f(x) = \infty$ $\qquad$ $\lim_{x\to\infty} g(x) = -\infty$

$\lim_{x\to-\infty} f(x) = -\infty$ $\qquad$ $\lim_{x\to-\infty} g(x) = \infty$

PART D: THE "ZOOM OUT" DOMINANCE PROPERTY

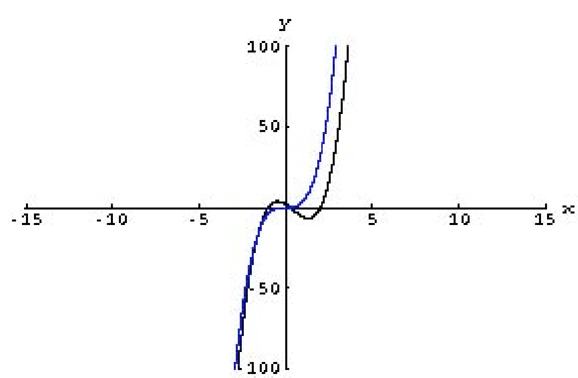
Example

The graph of $f(x) = 4x^3 - 5x^2 - 7x + 2$ is below.



Observe that $4x^3$ is the leading (i.e., highest-degree) term of $f(x)$.

If we "zoom out," we see that the graph looks similar to the graph for $4x^3$ (in blue below).



To determine the "long run" behavior of the graph of $f(x)$ as $x \to \infty$ and as $x \to -\infty$, it is sufficient to consider the graph for the leading term. (See p.123.)

Even if we don't know the graph of $f(x)$, we do know that the graph for $4x^3$ is a rising snake (in particular, a "stretched" version of the graph for x^3). We can conclude that:

$$\lim_{x \to \infty} f(x) = \infty$$
$$\lim_{x \to -\infty} f(x) = -\infty$$

"Zoom Out" Dominance Property of Leading Terms

> The leading term of a polynomial $f(x)$ increasingly dominates the other terms and increasingly determines the shape of the graph "in the long run" (as $x \to \infty$ and as $x \to -\infty$)

The lower-degree terms can put up a fight for part of the graph, and the struggle can lead to relative maximum and minimum points ("turning points (TPs)") along the graph. In the long run, however, the leading term dominates.

In Calculus: You will locate these turning points.

Note: The graph of $h(x) = (x+1)^4$ is on p.122. It's easiest to look at its graph as a translation of the x^4 bowl graph.

PART E: TURNING POINTS (TPs)

> The graph for a nonconstant n^{th}-degree polynomial $f(x)$ can have no more than $n-1$ TPs.

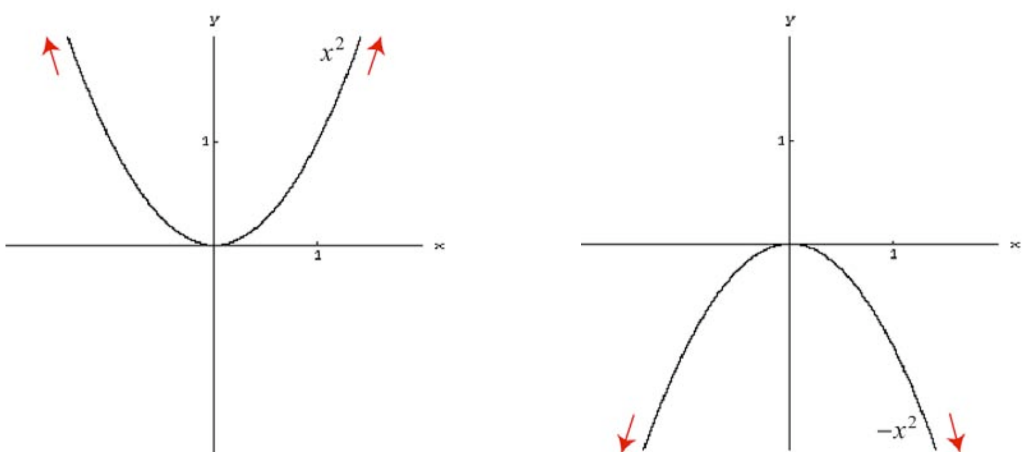
In Calculus: You will see why this is true.

Only high-degree polynomial functions can have very wavy graphs.

Even-Degree Case

If we trace a bowl graph from left to right, it "goes back to where it came from." In terms of "long run" behaviors, bowls "shoot off" in the same general direction: up or down.

Using notation, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are either both ∞ (as in the left graph) or both $-\infty$ (as in the right graph).



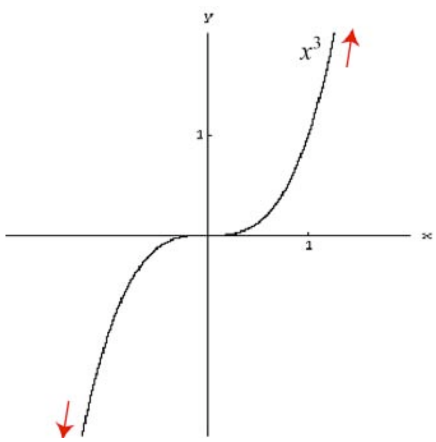
If we apply the "Zoom Out" Dominance Property, we see that this is true for **all** nonconstant even-degree $f(x)$. It must then be true that:

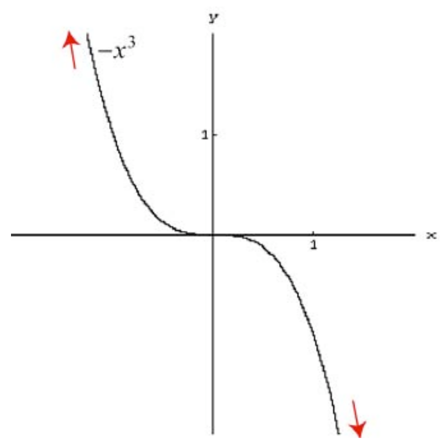
> The graph of a nonconstant **even**-degree polynomial $f(x)$ must have an **odd** number of TPs.

Odd-Degree Case

If we trace a snake graph from left to right, it "runs away from where it came from." In terms of "long run" behaviors, snakes "shoot off" in different directions: up **and** down.

Using notation, either $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$ must be ∞, and the other must be $-\infty$.





If we apply the "Zoom Out" Dominance Property, we see that this is true for **all** odd-degree $f(x)$. It must then be true that:

The graph of an **odd**-degree polynomial $f(x)$ must have an **even** number of TPs.

Another consequence:

An **odd**-degree polynomial $f(x)$ must have at least one real zero.

After all, its graph **must** have an x-intercept!

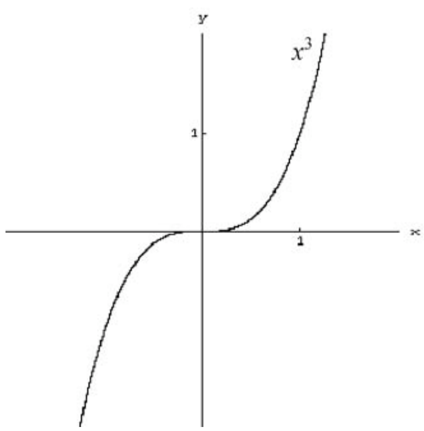
Example

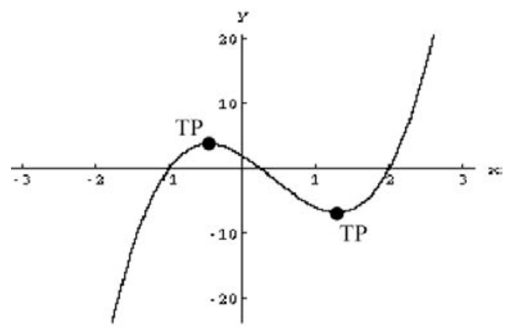
How many TPs can the graph of a 3rd-degree polynomial $f(x)$ have?

Solution

The degree is **odd**, so there must be an **even** number of TPs.
The degree is **3**, so $\left(\textbf{\# of TPs}\right) \le \mathbf{2}$.
Answer: The graph can have either **0 or 2** TPs.

Observe that the graph for x^3 on the left has 0 TPs, and the graph for $4x^3 - 5x^2 - 7x + 2$ on the right has 2 TPs.





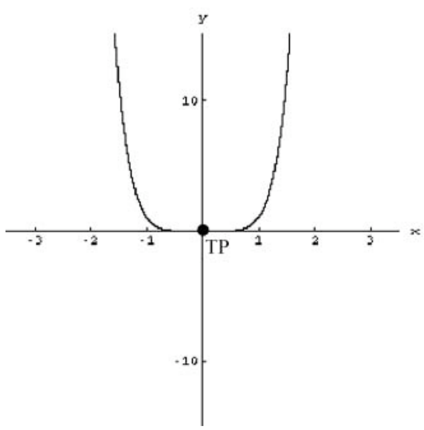
Example

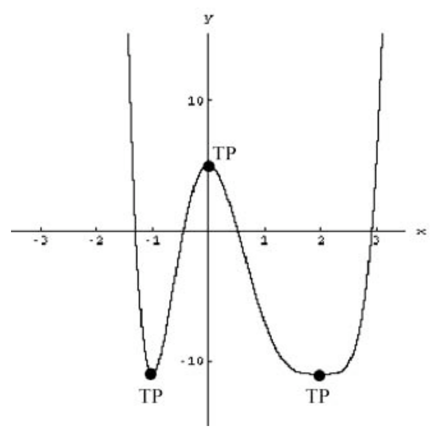
How many TPs can the graph of a 6^{th}-degree polynomial $f(x)$ have?

Solution

The degree is **even**, so there must be an **odd** number of TPs.
The degree is **6**, so $(\textbf{\# of TPs}) \leq \mathbf{5}$.
Answer: The graph can have **1, 3, or 5** TPs.

Observe that the graph for x^6 on the left has 1 TP, and the graph for $x^6 - 6x^5 + 9x^4 + 8x^3 - 24x^2 + 5$ on the right has 3 TPs.





Tip

Observe that, in both previous Examples, you start with $n-1$ and count down by twos. Stop before you reach negative numbers.

PART F: ZEROS AND THE FACTOR THEOREM

An n^{th}-degree polynomial $f(x)$ can have no more than n real zeros.

Factor Theorem

If $f(x)$ is a nonzero polynomial and k is a real number, then
k is a zero of f $\Leftrightarrow$ $(x-k)$ is a factor of $f(x)$.

Example

Find a 3rd-degree polynomial function that only has 4 and –1 as its zeros.

Solution

You may use anything of the form:

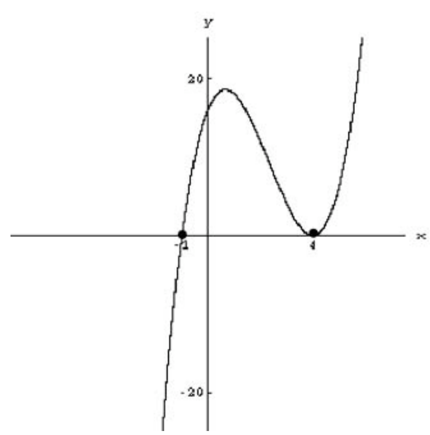
$$a(x-4)^2(x+1) \quad \text{or} \quad a(x-4)(x+1)^2,$$

where *a*, which turns out to be the leading coefficient of the expanded form, is any nonzero real number.

For example, we can use:

$$\begin{aligned} f(x) &= (x-4)^2(x+1) \\ &= x^3 - 7x^2 + 8x + 16 \end{aligned}$$

Its graph has 4 and –1 as its only *x*-intercepts:



Because the exponent on $(x-4)$ is 2, we say that 2 is the multiplicity of the zero "4." The x-intercept at 4 is a **TP**, because the multiplicity is **even** (and positive).

Technical Note: If we "zoom in" onto the x-intercept at 4, the graph appears to be almost symmetric about the line $x = 4$, due to the $(x-4)^2$ factor and the fact that, on a small interval containing 4, $(x+1)$ is almost a constant, 5.

PART G: THE INTERMEDIATE VALUE THEOREM, AND THE BISECTION METHOD FOR APPROXIMATING ZEROS

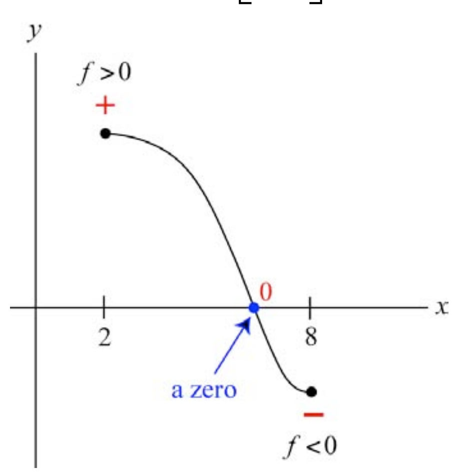
A dirty secret of mathematics is that we often have to use computer algorithms to help us approximate zeros of functions. While we do have (uglier) analogs of the Quadratic Formula for 3rd- and 4th-degree polynomial functions, it has actually been proven that there is no such formula for 5th- and higher-degree polynomial functions.

Polynomial functions are examples of continuous functions, whose graphs are unbroken in any way.

The Bisection Method for Approximating a Zero of a Continuous Function

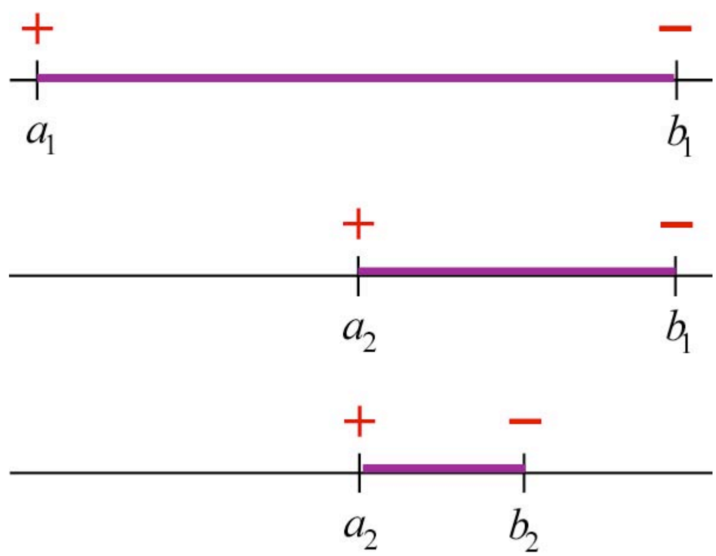
Try to find x-values a_1 and b_1 such that $f(a_1)$ and $f(b_1)$ have opposite signs. According to the Intermediate Value Theorem from Calculus, there must be a zero of f somewhere between a_1 and b_1. If $a_1 < b_1$, then we can call $[a_1, b_1]$ our "search interval."

For example, our search interval below is $[2, 8]$.



If either $f(a_1)$ or $f(b_1)$ is 0, then we have a zero of f, and we can either stop or try to approximate another zero.

If neither is 0, then we can take the midpoint of the search interval and find out what sign f is there (in red below). We can then shrink the search interval (in purple below) and repeat the process.



We repeat the process until we either find a zero, or until the search interval is small enough so that we can be happy with simply taking the midpoint of the interval as our approximation.

A key drawback to the Bisection Method is that, unless we manage to find n distinct real zeros of an n^{th}-degree polynomial $f(x)$, we may need other techniques to be sure that we have found **all** of the real zeros, if we are looking for all of them.

PART H: A CHECKLIST FOR GRAPHING POLYNOMIAL FUNCTIONS (BONUS TOPIC)

(You should know how to accurately graph constant, linear, and quadratic functions already.)

Remember that graphs of polynomial functions have no breaks, holes, cusps, or sharp corners (such as for $|x|$).

1) Find the *y*-intercept.

   It's the constant term of $f(x)$ in standard form.

2) Find the *x*-intercept(s), if any.

   In other words, find the real zeros of $f(x)$. Approximations may be necessary. We will discuss this further in Section 2.5.

3) Exploit symmetry, if possible.

   Is f even? Odd?

4) Use the "Zoom Out" Dominance Property.

   This determines the "long-run" behavior of the graph as $x \to \infty$ and as $x \to -\infty$.

   The book uses the "Leading Coefficient Test," which is essentially the same thing.

5) Find where $f(x) > 0$ and where $f(x) < 0$.

See p.126 for the Test Interval (or “Window”) Method.

A continuous function can only change sign at its zeros, so we know everything about the sign of f everywhere if we locate all of the zeros, break the x-axis (i.e., the real number line) into “test intervals” or “windows” by using the zeros as fence posts, and evaluate f at an x-value in each of the windows. The sign of f at a “test” x-value must be the sign of f all throughout the corresponding window.

The graph of f lies **above the *x*-axis** where $f(x) > 0$.

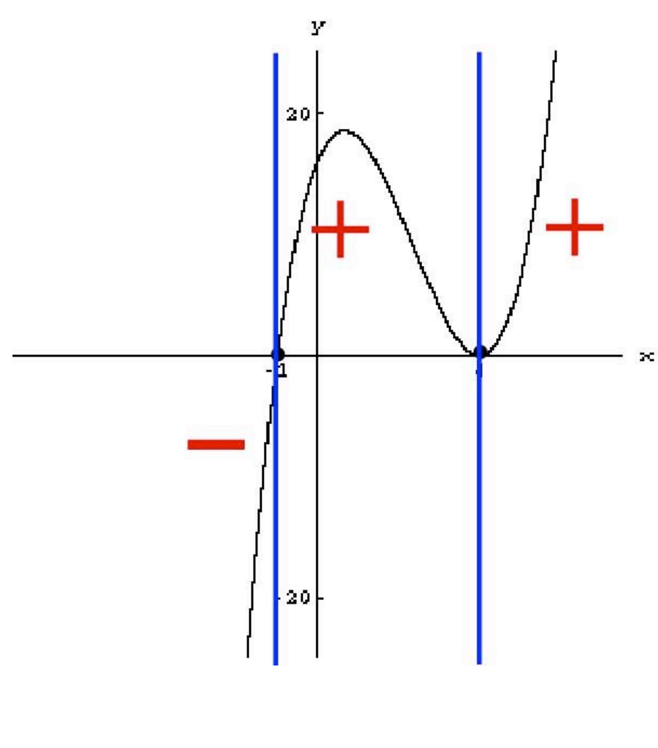
The graph of f has **an *x*-intercept** where $f(x) = 0$.

The graph of f lies **below the *x*-axis** where $f(x) < 0$.

Example

The graph below of $f(x) = (x-4)^2(x+1)$, or $x^3 - 7x^2 + 8x + 16$ appeared in Notes 2.19.

The signs of $f(x)$ are in red below. Window separators are in blue (they are not asymptotes).



If a zero of f has even multiplicity, then the graph has a TP there. If it has odd multiplicity, then it does not. You can avoid using the Test Interval (or "Window") Method if you know the complete factorization of $f(x)$ over the reals and use the "Zoom Out" Property.

6) Do point-plotting.

Do this as a last resort, or if you want a more accurate graph.

<u>In Calculus</u>: You will locate turning points (which are extremely helpful in drawing an accurate graph) and inflection points (points where the graph changes curvature from concave down to concave up or vice-versa). If you can locate all the turning points (if any), then you can find the **range** of the function graphically. (We know the domain of a polynomial function is always **R**.)

<u>Think About It</u>: What is the range of any odd-degree polynomial function? Can an even-degree polynomial function have the same range?