SECTION 2.4: COMPLEX NUMBERS

Let a, b, c, and d represent real numbers.

PART A: COMPLEX NUMBERS

i, the Imaginary Unit

We define: $i = \sqrt{-1}$.

$i^2 = -1$

If c is a positive real number $\left(c \in \mathbf{R}^+\right)$, then $\sqrt{-c} = i\sqrt{c}$.

Note: We often prefer writing $i\sqrt{c}$, as opposed to $\sqrt{c}i$, because we don't want to be confused about what is included in the radicand.

Examples

$$\sqrt{-15} = i\sqrt{15}$$

$$\sqrt{-16} = i\sqrt{16} = 4i$$

$$\sqrt{-18} = i\sqrt{18} = i\sqrt{9 \cdot 2} = 3i\sqrt{2}$$

Here, we used the fact that 9 is the largest perfect square that divides 18 evenly. The 9 "comes out" of the square root radical as $\sqrt{9}$, or 3.

Standard Form for a Complex Number

$a + bi$, where a and b are real numbers $\left(a, b \in \mathbf{R}\right)$.

a is the real part;
b or bi is the imaginary part.

C is the set of all complex numbers, which includes all real numbers.
In other words, $\mathbf{R} \subseteq \mathbf{C}$.

Examples

2, $3i$, and $2+3i$ are all complex numbers.

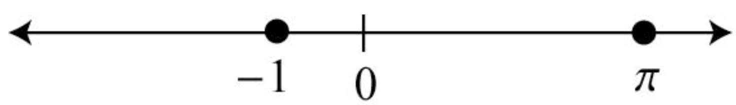
2 is also a real number.

$3i$ is called a pure imaginary number, because $a=0$ and $b\neq 0$ here.

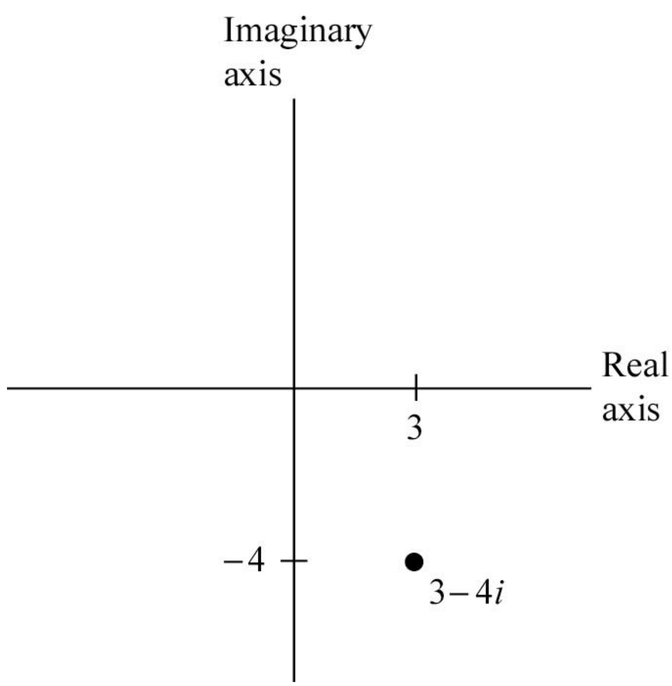
$2+3i$ is called an imaginary number, because it is a nonreal complex number.

PART B: THE COMPLEX PLANE

The real number line (below) exhibits a linear ordering of the real numbers.
In other words, if c and d are real numbers, then exactly one of the following must be true: $c<d$, $c>d$, or $c=d$.

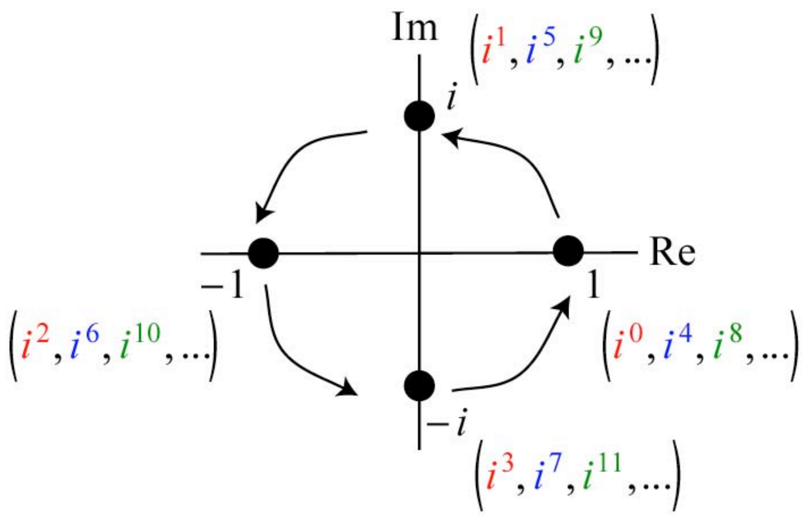


The complex plane (below) exhibits no such linear ordering of the complex numbers.
We plot the number $a+bi$ as the point (a,b).



PART C: POWERS OF *i*

$i^0 = 1$	$i^4 = \left(i^2\right)^2 = \left(-1\right)^2 = 1$	
$i^1 = i$	$i^5 = i^4 i = \left(1\right)\left(i\right) = i$	
$i^2 = -1$	$i^6 = i^4 i^2 = \left(1\right)\left(-1\right) = -1$	$\cdots$
$i^3 = i^2 i = \left(-1\right)i = -i$	$i^7 = i^4 i^3 = \left(1\right)\left(-i\right) = -i$	



Technical Note: In general, it is true that when a complex number is multiplied by *i*, the corresponding point in the complex plane is rotated 90° counterclockwise about the origin.

$$i^n = i^{\text{remainder when } n \text{ is divided by } 4} \qquad \left(n \in \mathbf{Z}^+\right)$$

Know the first column of the table at the top of this page!

Examples:

Simplify the following expressions:

$i^{100} = i^0 = 1$. Note that 100 is a multiple of 4.

$i^{83} = i^3 = -i$

PART D: ADDING, SUBTRACTING, AND MULTIPLYING COMPLEX NUMBERS

Steps

Step 1: Simplify all radicals.

Step 2: Think of i as x. Add, subtract, and multiply as usual.
Warning: However, simplify all powers of i (except i, itself).

Step 3: Simplify down to the $a+bi$ standard form $\left(a, b \in \mathbf{R}\right)$.

Example

Simplify $\sqrt{-9}\sqrt{-4}$.

Solution

Warning: Do Step 1 first! The Product Rule for Radicals does **not** apply in this situation. $\sqrt{-9}\sqrt{-4}$ is **not** equivalent to $\sqrt{(-9)(-4)}$, which would have given us $\sqrt{36}=6$.

$$\begin{aligned}\sqrt{-9}\sqrt{-4} &= (3i)(2i)\\ &= 6i^2\\ &= 6(-1)\\ &= \mathbf{-6}\end{aligned}$$

Example

Simplify $(2+3i)(4+i)$.

Solution

$$\begin{aligned}(2+3i)(4+i) &= 8+2i+12i+3i^2\\ &= 8+14i+3(-1)\\ &= 8+14i-3\\ &= \mathbf{5+14}\boldsymbol{i}\end{aligned}$$

PART E: COMPLEX CONJUGATES

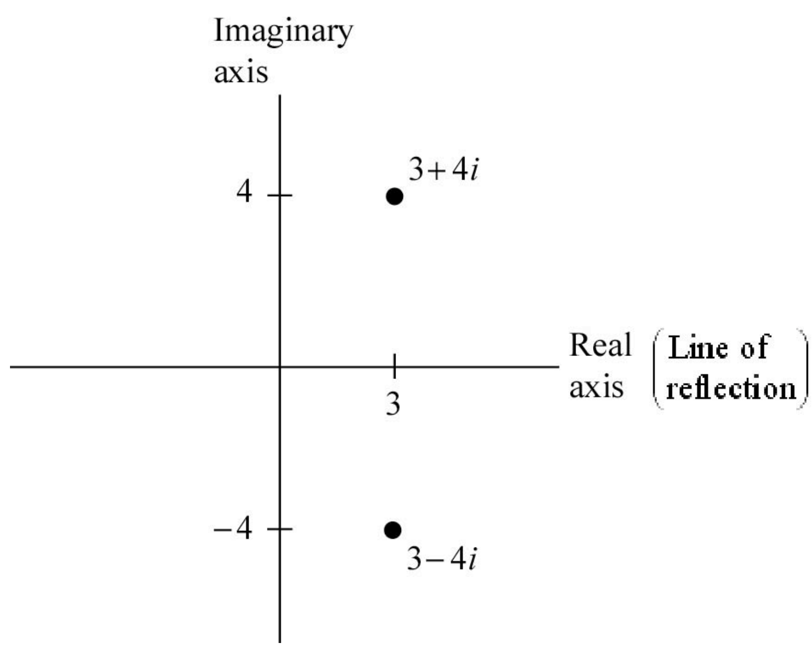
Let z be a complex number $\left(z \in \mathbf{C}\right)$.

$\bar{z}$ is the complex conjugate of z.

If $z = a + bi$, then $\bar{z} = a - bi$. $\left(a, b \in \mathbf{R}\right)$

Example

z	$\bar{z}$	**Think**
$3-4i$	$3+4i$	(See graph below)
$-1+2i$	$-1-2i$	
5	5	$5=5+0i$
$-4i$	$4i$	$-4i=0-4i$



Note: You may recall that the conjugate of $3+\sqrt{2}$, for example, is $3-\sqrt{2}$.

If z is a complex number, then $z\bar{z}$ is a real number.
That is, $\left(z \in \mathbf{C}\right) \Rightarrow \left(z\bar{z} \in \mathbf{R}\right)$

In fact, if $z = a + bi \ \left(a, b \in \mathbf{R}\right)$, then:

$$
\begin{aligned}
z\bar{z} &= \left(a+bi\right)\left(a-bi\right) \\
&= a^2 - \left(bi\right)^2 \\
&= a^2 - b^2 i^2 \\
&= a^2 - b^2\left(-1\right) \\
&= a^2 + b^2, \text{ which is a real number}
\end{aligned}
$$

From the above work, we also get that $a^2 + b^2 = \left(a+bi\right)\left(a-bi\right)$.
This will share the form of a key factoring rule we will use in Section 2.5.

PART F: DIVIDING COMPLEX NUMBERS

Goal: Express the quotient $\frac{z_1}{z_2}$ in $a+bi$ standard form $\left(z_1, z_2 \in \mathbf{C};\ a, b \in \mathbf{R}\right)$.

We need to rationalize the denominator. (Actually, "real"-ize may be more appropriate, since we may end up with irrational denominators.)

Steps

Step 1: Simplify all radicals.

Step 2:

If $z_2 = di$ $\left(d \in \mathbf{R}\right)$, a pure imaginary number, then multiply z_1 and z_2 by i or $-i$.

If $z_2 = c + di$ $\left(c, d \in \mathbf{R}\right)$, then multiply z_1 and z_2 by $\overline{z_2}$, the complex conjugate of the denominator (namely, $c - di$).

Step 3: Simplify down to the $a+bi$ standard form $\left(a, b \in \mathbf{R}\right)$.

Example

Simplify $\dfrac{4-i}{3i}$.

Solution

$$\begin{aligned}\frac{4-i}{3i} &= \frac{(4-i)}{3i}\cdot\frac{i}{i}\\ &= \frac{4i-i^2}{3i^2}\\ &= \frac{4i-(-1)}{3(-1)}\\ &= \frac{4i+1}{-3}\\ &= -\frac{4}{3}i-\frac{1}{3}\\ &= \boldsymbol{-\frac{1}{3}-\frac{4}{3}i}\end{aligned}$$

Warning: The last step above is necessary to obtain "standard form."

Example

Simplify $\dfrac{\sqrt{-16}}{4+\sqrt{-9}}$.

Solution

$$\frac{\sqrt{-16}}{4+\sqrt{-9}} = \frac{4i}{4+3i}$$

$$= \frac{4i}{(4+3i)} \cdot \frac{(4-3i)}{(4-3i)}$$

Simplify $(4+3i)(4-3i)$:

Method 1: Use: $(a+bi)(a-bi) = a^2 + b^2$. (Notes 2.40)

$$(4+3i)(4-3i) = (4)^2 + (3)^2 = 25$$

Method 2: Use: $(A+B)(A-B) = A^2 - B^2$.

$$\begin{aligned}(4+3i)(4-3i) &= (4)^2 - (3i)^2 \\ &= 16 - 9i^2 \\ &= 16 - 9(-1) \\ &= 16 + 9 \\ &= 25\end{aligned}$$

Method 3: Use FOIL.

$$\begin{aligned}(4+3i)(4-3i) &= 16 - 12i + 12i - 9i^2 \\ &= 16 - 9i^2 \\ &\vdots \quad \text{(See work in Method 2.)} \\ &= 25\end{aligned}$$

$$= \frac{4i}{(4+3i)} \cdot \frac{(4-3i)}{(4-3i)} \qquad (\text{Reminder})$$

$$= \frac{16i - 12i^2}{25}$$

$$= \frac{16i - 12(-1)}{25}$$

$$= \frac{16i + 12}{25}$$

$$= \frac{16}{25}i + \frac{12}{25}$$

$$= \mathbf{\frac{12}{25} + \frac{16}{25}\boldsymbol{i}}$$

PART G: COMPLEX ZEROS OF FUNCTIONS

Example

Find all complex zeros (potentially including real zeros) of $f(x) = x^2 + 9$.

Solution (using the Square Root Method)

$$\begin{aligned} x^2 + 9 &= 0 \\ x^2 &= -9 \\ x &= \pm\sqrt{-9} \\ x &= \pm \mathbf{3}\boldsymbol{i} \end{aligned}$$

Example

Find all complex zeros of $f(x) = 2x^2 - x + 3$.

Solution (using the QF)

We must solve $2x^2 - x + 3 = 0$. $(a = 2,\ b = -1,\ c = 3)$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(3)}}{2(2)} \\ &= \frac{1 \pm \sqrt{-23}}{4} \\ &= \frac{1 \pm i\sqrt{23}}{4} \\ &= \frac{1}{4} \pm \frac{i\sqrt{23}}{4} \\ &= \mathbf{\frac{1}{4} \pm \frac{\sqrt{23}}{4}}\boldsymbol{i} \end{aligned}$$

Warning: The last step above is necessary to obtain "standard form."