

## SECTION 2.4: COMPLEX NUMBERS

Let  $a$ ,  $b$ ,  $c$ , and  $d$  represent real numbers.

### PART A: COMPLEX NUMBERS

$i$ , the Imaginary Unit

We define:  $i = \sqrt{-1}$ .

$$i^2 = -1$$

If  $c$  is a positive real number ( $c \in \mathbf{R}^+$ ), then  $\sqrt{-c} = i\sqrt{c}$ .

Note: We often prefer writing  $i\sqrt{c}$ , as opposed to  $\sqrt{ci}$ , because we don't want to be confused about what is included in the radicand.

Examples

$$\sqrt{-15} = i\sqrt{15}$$

$$\sqrt{-16} = i\sqrt{16} = 4i$$

$$\sqrt{-18} = i\sqrt{18} = i\sqrt{9 \cdot 2} = 3i\sqrt{2}$$

Here, we used the fact that 9 is the largest perfect square that divides 18 evenly. The 9 “comes out” of the square root radical as  $\sqrt{9}$ , or 3.

Standard Form for a Complex Number

$a + bi$ , where  $a$  and  $b$  are real numbers ( $a, b \in \mathbf{R}$ ).

$a$  is the real part;  
 $b$  or  $bi$  is the imaginary part.

$\mathbf{C}$  is the set of all complex numbers, which includes all real numbers. In other words,  $\mathbf{R} \subseteq \mathbf{C}$ .

Examples

2,  $3i$ , and  $2 + 3i$  are all complex numbers.

2 is also a real number.

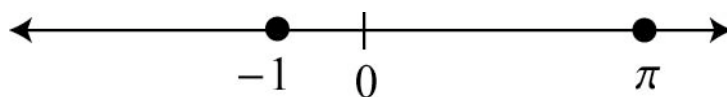
$3i$  is called a pure imaginary number, because  $a = 0$  and  $b \neq 0$  here.

$2 + 3i$  is called an imaginary number, because it is a nonreal complex number.

**PART B: THE COMPLEX PLANE**

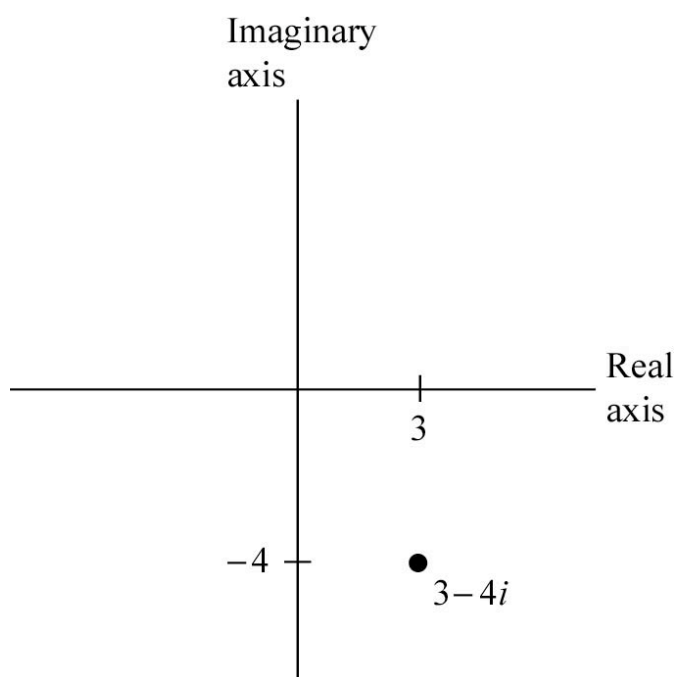
The real number line (below) exhibits a linear ordering of the real numbers.

In other words, if  $c$  and  $d$  are real numbers, then exactly one of the following must be true:  $c < d$ ,  $c > d$ , or  $c = d$ .



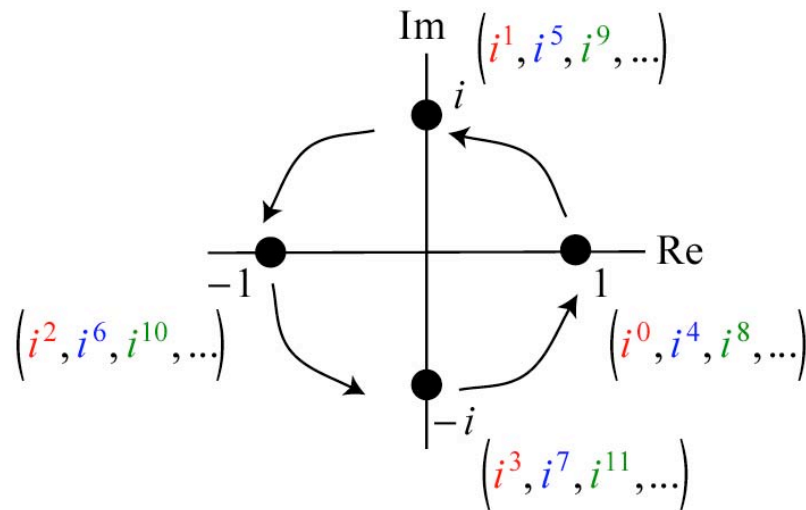
The complex plane (below) exhibits no such linear ordering of the complex numbers.

We plot the number  $a + bi$  as the point  $(a, b)$ .



**PART C: POWERS OF  $i$** 

$$\begin{array}{llll}
 i^0 & = 1 & i^4 = (i^2)^2 = (-1)^2 = 1 & \\
 i^1 & = i & i^5 = i^4 i = (1)(i) = i & \\
 i^2 & = -1 & i^6 = i^4 i^2 = (1)(-1) = -1 & \dots \\
 i^3 = i^2 i = (-1)i = -i & & i^7 = i^4 i^3 = (1)(-i) = -i & 
 \end{array}$$



**Technical Note:** In general, it is true that when a complex number is multiplied by  $i$ , the corresponding point in the complex plane is rotated  $90^\circ$  counterclockwise about the origin.

$$i^n = i^{\text{remainder when } n \text{ is divided by } 4} \quad (n \in \mathbf{Z}^+)$$

Know the first column of the table at the top of this page!

Examples:

Simplify the following expressions:

$$i^{100} = i^0 = 1. \text{ Note that } 100 \text{ is a multiple of } 4.$$

$$i^{83} = i^3 = -i$$

## PART D: ADDING, SUBTRACTING, AND MULTIPLYING COMPLEX NUMBERS

### Steps

Step 1: Simplify all radicals.

Step 2: Think of  $i$  as  $x$ . Add, subtract, and multiply as usual.

Warning: However, simplify all powers of  $i$  (except  $i$ , itself).

Step 3: Simplify down to the  $a + bi$  standard form ( $a, b \in \mathbf{R}$ ).

### Example

Simplify  $\sqrt{-9}\sqrt{-4}$ .

### Solution

Warning: Do Step 1 first! The Product Rule for Radicals does **not** apply in this situation.  $\sqrt{-9}\sqrt{-4}$  is **not** equivalent to  $\sqrt{(-9)(-4)}$ , which would have given us  $\sqrt{36} = 6$ .

$$\begin{aligned}\sqrt{-9}\sqrt{-4} &= (3i)(2i) \\ &= 6i^2 \\ &= 6(-1) \\ &= -6\end{aligned}$$

### Example

Simplify  $(2 + 3i)(4 + i)$ .

### Solution

$$\begin{aligned}(2 + 3i)(4 + i) &= 8 + 2i + 12i + 3i^2 \\ &= 8 + 14i + 3(-1) \\ &= 8 + 14i - 3 \\ &= \mathbf{5 + 14i}\end{aligned}$$

**PART E: COMPLEX CONJUGATES**

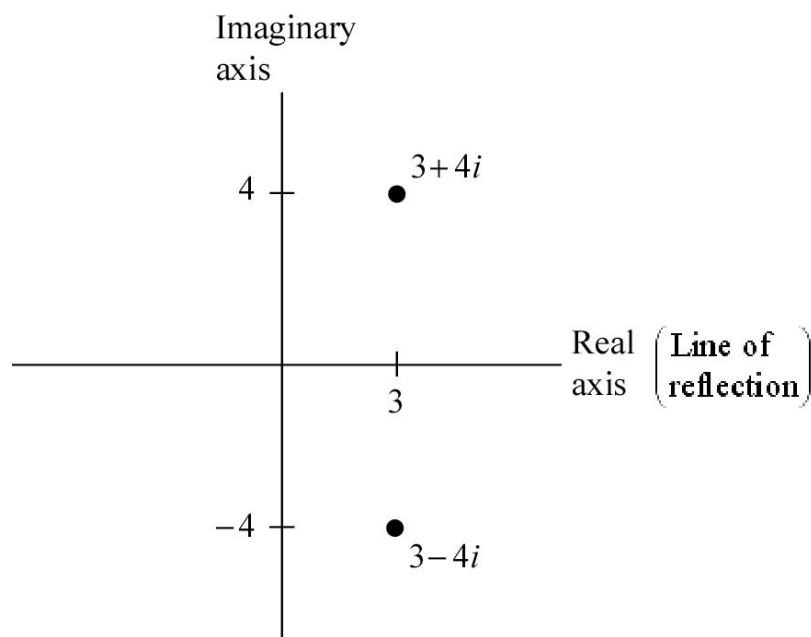
Let  $z$  be a complex number ( $z \in \mathbf{C}$ ).

$\bar{z}$  is the complex conjugate of  $z$ .

If  $z = a + bi$ , then  $\bar{z} = a - bi$ . ( $a, b \in \mathbf{R}$ )

**Example**

$z$	$\bar{z}$	<b>Think</b>
$3 - 4i$	$3 + 4i$	(See graph below)
$-1 + 2i$	$-1 - 2i$	
$5$	$5$	$5 = 5 + 0i$
$-4i$	$4i$	$-4i = 0 - 4i$



Note: You may recall that the conjugate of  $3 + \sqrt{2}$ , for example, is  $3 - \sqrt{2}$ .

If  $z$  is a complex number, then  $z\bar{z}$  is a real number.  
That is,  $(z \in \mathbf{C}) \Rightarrow (z\bar{z} \in \mathbf{R})$

In fact, if  $z = a + bi$  ( $a, b \in \mathbf{R}$ ), then:

$$\begin{aligned}z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 - (bi)^2 \\ &= a^2 - b^2i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2, \text{ which is a real number}\end{aligned}$$

From the above work, we also get that  $a^2 + b^2 = (a + bi)(a - bi)$ .

This will share the form of a key factoring rule we will use in [Section 2.5](#).

**PART F: DIVIDING COMPLEX NUMBERS**

Goal: Express the quotient  $\frac{z_1}{z_2}$  in  $a + bi$  standard form ( $z_1, z_2 \in \mathbf{C}$ ;  $a, b \in \mathbf{R}$ ).

We need to rationalize the denominator. (Actually, “real”-ize may be more appropriate, since we may end up with irrational denominators.)

Steps

Step 1: Simplify all radicals.

Step 2:

If  $z_2 = di$  ( $d \in \mathbf{R}$ ), a pure imaginary number, then multiply  $z_1$  and  $z_2$  by  $i$  or  $-i$ .

If  $z_2 = c + di$  ( $c, d \in \mathbf{R}$ ), then multiply  $z_1$  and  $z_2$  by  $\overline{z_2}$ , the complex conjugate of the denominator (namely,  $c - di$ ).

Step 3: Simplify down to the  $a + bi$  standard form ( $a, b \in \mathbf{R}$ ).

Example

Simplify  $\frac{4-i}{3i}$ .

Solution

$$\begin{aligned}\frac{4-i}{3i} &= \frac{(4-i)}{3i} \cdot \frac{i}{i} \\ &= \frac{4i-i^2}{3i^2} \\ &= \frac{4i-(-1)}{3(-1)} \\ &= \frac{4i+1}{-3} \\ &= -\frac{4}{3}i - \frac{1}{3} \\ &= -\frac{1}{3} - \frac{4}{3}i\end{aligned}$$

Warning: The last step above is necessary to obtain “standard form.”

Example

Simplify  $\frac{\sqrt{-16}}{4 + \sqrt{-9}}$ .

Solution

$$\begin{aligned}\frac{\sqrt{-16}}{4 + \sqrt{-9}} &= \frac{4i}{4 + 3i} \\ &= \frac{4i}{(4 + 3i)} \cdot \frac{(4 - 3i)}{(4 - 3i)}\end{aligned}$$

Simplify  $(4 + 3i)(4 - 3i)$ :

Method 1: Use:  $(a + bi)(a - bi) = a^2 + b^2$ . (Notes 2.40)

$$(4 + 3i)(4 - 3i) = (4)^2 + (3)^2 = 25$$

Method 2: Use:  $(A + B)(A - B) = A^2 - B^2$ .

$$\begin{aligned}(4 + 3i)(4 - 3i) &= (4)^2 - (3i)^2 \\ &= 16 - 9i^2 \\ &= 16 - 9(-1) \\ &= 16 + 9 \\ &= 25\end{aligned}$$

Method 3: Use FOIL.

$$\begin{aligned}(4 + 3i)(4 - 3i) &= 16 - 12i + 12i - 9i^2 \\ &= 16 - 9i^2 \\ &\quad \vdots \text{ (See work in Method 2.)} \\ &= 25\end{aligned}$$

$$\begin{aligned} &= \frac{4i}{(4+3i)} \cdot \frac{(4-3i)}{(4-3i)} && \text{(Reminder)} \\ &= \frac{16i - 12i^2}{25} \\ &= \frac{16i - 12(-1)}{25} \\ &= \frac{16i + 12}{25} \\ &= \frac{16}{25}i + \frac{12}{25} \\ &= \frac{12}{25} + \frac{16}{25}i \end{aligned}$$

**PART G: COMPLEX ZEROS OF FUNCTIONS**Example

Find all complex zeros (potentially including real zeros) of  $f(x) = x^2 + 9$ .

Solution (using the Square Root Method)

$$\begin{aligned}x^2 + 9 &= 0 \\x^2 &= -9 \\x &= \pm\sqrt{-9} \\x &= \pm 3i\end{aligned}$$

Example

Find all complex zeros of  $f(x) = 2x^2 - x + 3$ .

Solution (using the QF)

We must solve  $2x^2 - x + 3 = 0$ . ( $a = 2$ ,  $b = -1$ ,  $c = 3$ )

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(3)}}{2(2)} \\&= \frac{1 \pm \sqrt{-23}}{4} \\&= \frac{1 \pm i\sqrt{23}}{4} \\&= \frac{1}{4} \pm \frac{i\sqrt{23}}{4} \\&= \frac{1}{4} \pm \frac{\sqrt{23}}{4}i\end{aligned}$$

**Warning:** The last step above is necessary to obtain “standard form.”