SECTION 2.5: FINDING ZEROS OF POLYNOMIAL FUNCTIONS

Assume $f(x)$ is a nonconstant polynomial with real coefficients written in standard form.

PART A: TECHNIQUES WE HAVE ALREADY SEEN

Refer to:

Notes 1.31 to 1.35
Section A.5 in the book
Notes 2.45

	Refer to
1) Factoring	(Notes 1.33)
2) Methods for Dealing with Quadratic Functions	(Book Section A.5: pp.A49-51)
a) Square Root Method	(Notes 1.31, 2.45)
b) Factoring	(Notes 1.33)
c) QF	(Notes 1.34, 2.45)
d) CTS (Completing the Square)	(Book Section A.5: p.A49)
3) Bisection Method (for Approximating Zeros)	(Notes 2.20 to 2.21)
4) Synthetic Division and the Remainder Theorem (for Verifying Zeros)	(Notes 2.33)

PART B: RATIONAL ZERO TEST

Rational Zero Test (or Rational Roots Theorem)

Let $f(x)$ be a polynomial with integer (i.e., only integer) coefficients written in standard form:

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$
$$\left(\text{each constant } a_i \in \mathbf{Z};\ a_n \neq 0;\ a_0 \neq 0;\ n \in \mathbf{Z}^+\right)$$

If $f(x)$ has rational zeros, they must be in the list of $\pm\frac{p}{q}$ candidates, where:

p is a factor of a_0, the constant term, and
q is a factor of a_n, the leading coefficient.

Note: We require $a_0 \neq 0$. If $a_0 = 0$, try factoring out the GCF first.

Example

Factor $f(x) = 4x^3 - 5x^2 - 7x + 2$ completely, and find **all** of its real zeros.

Solution

Since the $\text{GCF} = 1$, and Factoring by Grouping does not seem to help, we resort to using the Rational Zero Test. We will now list the candidates for possible rational zeros of $f(x)$.

p (factors of the constant term, 2): $\pm 1,\ \pm 2$
q (factors of the leading coefficient, 4): $\pm 1,\ \pm 2,\ \pm 4$

Note: You may omit the $\pm$ symbols above if you use them below.

List of $\pm\frac{p}{q}$ candidates:

$$\pm\frac{1}{1},\ \pm\frac{1}{2},\ \pm\frac{1}{4},\ \pm\frac{2}{1},\ \pm\frac{2}{2},\ \pm\frac{2}{4}$$

Simplified:

$$\pm 1,\ \pm\frac{1}{2},\ \pm\frac{1}{4},\ \pm 2,\ \underbrace{\pm 1,\ \pm\frac{1}{2}}_{\text{Redundant}}$$

Use Synthetic Division to divide $f(x)$ by $(x-k)$, where k is one of our rational candidates. Remember that the following are equivalent for a nonzero polynomial $f(x)$ and a real number k:

$(x-k)$ is a factor of $f(x)$ $\Leftrightarrow$
k is a zero of $f(x)$ $(\text{i.e.,} f(k)=0)$ $\Leftrightarrow$
We get a 0 remainder in the Synthetic Division process.

The first $\Leftrightarrow$ is the Factor Theorem, and the second $\Leftrightarrow$ comes from the Remainder Theorem. See Notes on Section 2.3: 2.32-2.34.

We use trial-and-error and proceed through our list of candidates (k) for rational zeros: $\pm 1,\ \pm\frac{1}{2},\ \pm\frac{1}{4},\ \pm 2$

Let's try $k=1$.

<u>Method 1</u>

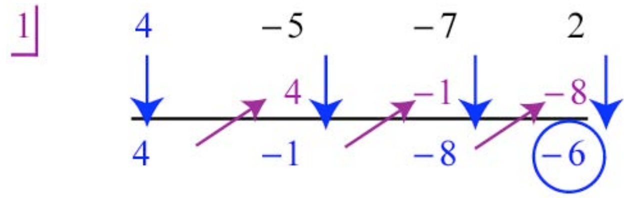
We may directly evaluate $f(1)$ and see if it is 0.

$$f(x)=4x^3-5x^2-7x+2$$
$$f(1)=4(1)^3-5(1)^2-7(1)+2$$
$$=-6 \quad (\neq 0)$$

Therefore, 1 is **not** a zero of $f(x)$.

Method 2

We may also use the Synthetic Division process and see if we get a 0 remainder.



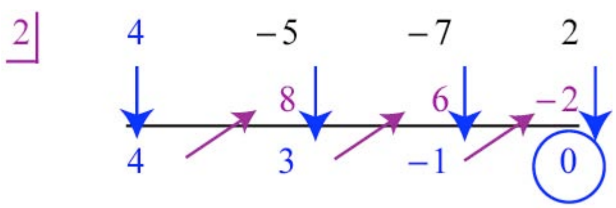
We do **not** get a 0 remainder, so 1 is **not** a zero of $f(x)$.

Method 3

Observe from both previous methods that we can compute $f(1)$ by simply adding up the coefficients of $f(x)$ in standard form. This does not work in general for other values of k, though.

Let's try $k = 2$.

Let's use the Synthetic Division / Remainder Theorem method:



We **do** get a 0 remainder, so 2 **is** a zero of $f(x)$.

This turns out to be the key that cracks the whole problem. Incidentally, this is the same $f(x)$ that we saw in Notes 2.33-2.35. Now we know how our "little bird" got its info!

By the Factor Theorem, $(x-2)$ must be a factor of $f(x)$.

We can find $q(x)$, the other (quadratic) factor, by using the last row of the table.

$$f(x)=(x-2)\cdot\left(4x^2+3x-1\right)$$

Factor $q(x)$ completely over the reals:

$$\boldsymbol{f(x)=(x-2)(4x-1)(x+1)}$$

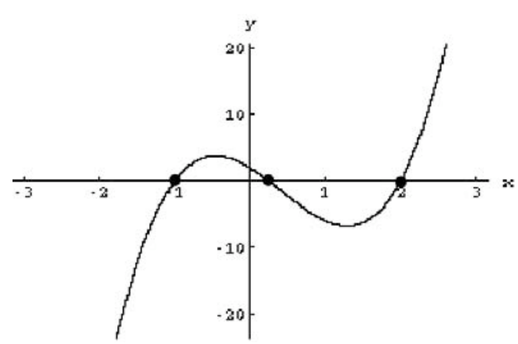
> Remember that we always know how to break down a quadratic (use the QF if you have to), although its zeros may or may not be real. We no longer have to rely on rational zeros at this point.

The zeros of $f(x)$ are the zeros of these factors:

$$\mathbf{2,\ \frac{1}{4},\ -1}$$

Observe that all three are rational and appeared in our list of candidates for rational zeros. Any one of these three could have been used to start cracking the problem.

Below is a graph of $f(x)=4x^3-5x^2-7x+2$. Where are the *x*-intercepts?



<u>Note</u>: If we can get a graph of $f(x)$ beforehand, then we may be able to choose our guesses for rational zeros more wisely.

Warning: Remember that the template for our list of candidates is:

$$\pm \frac{\text{factor of constant term}}{\text{factor of leading coefficient}},$$

not the reciprocal. One way to remember which way the template goes is to use a simple example such as $x-2$. The list must be $\pm 1, \pm 2$ and not $\pm 1, \pm \frac{1}{2}$.

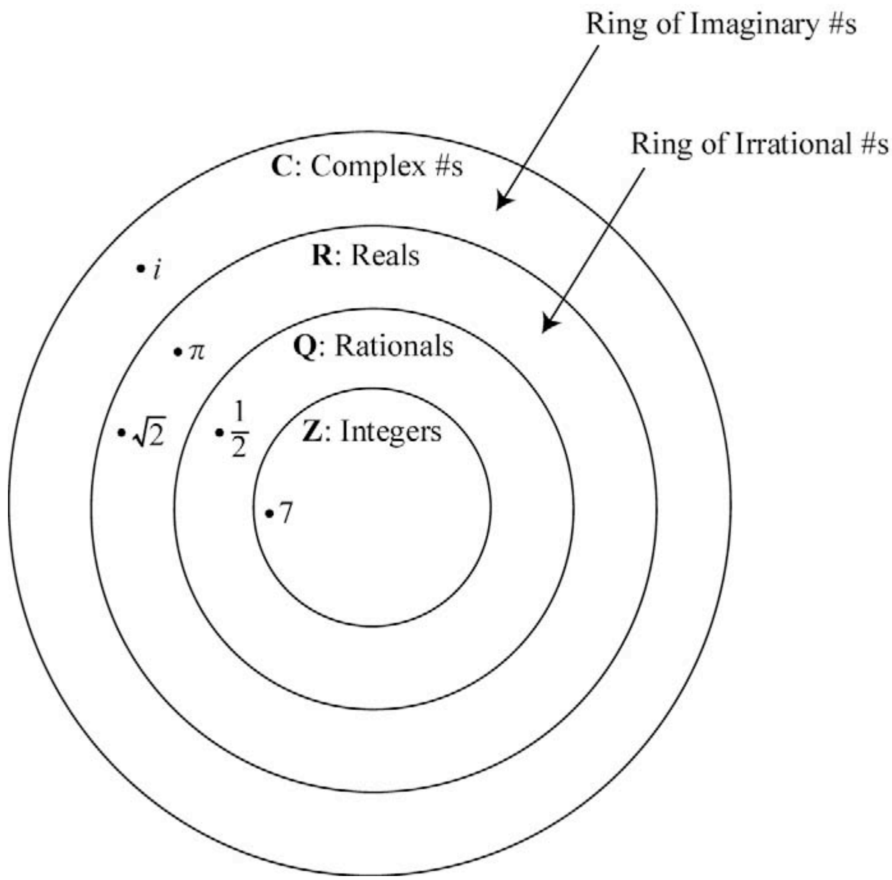
Note: If none of our rational candidates work, then $f(x)$ has no rational zeros.
For example, $x^2 - 3$ has no rational zeros.

Note: Synthetic Division may be applied repeatedly. It may be a good idea to revise the list of rational candidates before each new application (the textbook does not do this). A candidate that "worked" as a zero before may work again (in which case it is a repeated zero), but a candidate that "failed" before will never work. You may want to stop using Synthetic Division when you get down to a quadratic, or when you get down to something you think you can factor, such as $x^4 - 1$.

Note: Synthetic Division may be used when we are dealing with imaginary zeros and imaginary coefficients. This is reflected in the students' Study and Solutions Guide.

PART C: FACTORING OVER VARIOUS SETS

Recall our Venn diagram from Notes P.03:



Note: **Z** comes from "Zahlen," the German word for "integer."
Q comes from "Quotient."

Factoring over **Z** (the Integers)

Example

In our Example in Part B, we factored:

$$4x^3 - 5x^2 - 7x + 2 = (x-2)(4x-1)(x+1)$$

This is an example of factoring over **Z**, because we only use integers as coefficients (including constant terms within factors).

Factoring over **Q** (the Rationals)

Example

Let's factor a 4 out of the second factor in the previous Example.

$$4x^3 - 5x^2 - 7x + 2 = (x-2)(4x-1)(x+1) \quad \leftarrow \text{Factored over } \mathbf{Z}$$
$$= 4(x-2)\left(x - \frac{1}{4}\right)(x+1) \quad \leftarrow \text{Factored over } \mathbf{Q}$$

The "Factored over **Z**" expression is also an example of factoring over **Q**, but this new factorization over **Q** immediately identifies $\frac{1}{4}$ as a zero.

Factoring over **R** (the Reals)

Example

$x^2 - 3$ is prime (or irreducible) over **Z** and **Q**; it cannot be factored further (nontrivially; breaking out a 1 or a -1 doesn't count) using only integer or rational coefficients. However, it can be factored over **R**.

$$x^2 - 3 = \left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right)$$

Note that $-\sqrt{3}$ and $\sqrt{3}$ are immediately identified as zeros.

Factoring over **C** (the Complex Numbers)

Recall our work from Notes 2.40. We found that:
$a^2 + b^2 = (a+bi)(a-bi)$, where a and b were real numbers.
However, this form is also appropriate if a and/or b represent variable expressions.

Example

$x^2 + 9$ is prime (or irreducible) over **R**. However, it can be factored over **C**.

$$x^2 + 9 = (x+3i)(x-3i)$$

Note that $-3i$ and $3i$ are immediately identified as zeros here.

Example

Factor (i.e., factor completely) $x^5 + x^3 - 6x$ over **R** and find all of its real zeros.

Solution

First factor out the GCF, x.

$$x^5 + x^3 - 6x = x\left(x^4 + x^2 - 6\right)$$

The second factor is in Quadratic Form, because it is of the form $u^2 + u - 6$, where $u = x^2$. How do we know $x^4 + x^2 - 6$ is in Quadratic Form? Observe that the exponent on x in the first term is twice the exponent on x in the second term, and the third term is a constant.

Substitute $u = x^2$ (optional, but it may help):

$$\begin{aligned} x^5 + x^3 - 6x &= x\left(x^4 + x^2 - 6\right) \\ &= x\left(u^2 + u - 6\right) \\ &= x(u+3)(u-2) \\ &= x\left(x^2 + 3\right)\left(x^2 - 2\right) \qquad \leftarrow \text{Substitute back.} \end{aligned}$$

We have factored completely over **Z** (and **Q**).
Let us now factor completely over **R**.

$$x^5 + x^3 - 6x = x\left(x^2+3\right)\left(x^2-2\right) \quad \leftarrow \text{Reminder}$$
$$= \boldsymbol{x\left(x^2+3\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)} \quad \leftarrow \text{Factored over } \mathbf{R}$$

$\left(x^2+3\right)$ is a quadratic that is irreducible over the reals. It therefore yields no real zeros. (We need more theorems to show this.)

From the other three factors, we obtain $\mathbf{0, -\sqrt{2}, \text{and } \sqrt{2}}$ as real zeros.

<u>Example</u>

Factor $f\left(x\right) = x^5 + x^3 - 6x$ over **C** and find all of its real zeros.

<u>Solution</u>

We continue with our work from the previous Example.

We can factor $\left(x^2+3\right)$ further over **C**.

$$x^5 + x^3 - 6x = x\left(x^2+3\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right) \quad \leftarrow \text{Factored over } \mathbf{R}$$
$$= \boldsymbol{x\left(x+i\sqrt{3}\right)\left(x-i\sqrt{3}\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)} \quad \leftarrow \text{Factored over } \mathbf{C}$$

We immediately obtain the five complex zeros of $f\left(x\right)$:

$$\boldsymbol{0,\ -i\sqrt{3},\ i\sqrt{3},\ -\sqrt{2},\ \text{and } \sqrt{2}}$$

PART D: COMPLEX CONJUGATE PAIRS OF ZEROS

Complex Conjugate Pairs Theorem

Let $a, b \in \mathbf{R}$.

If $f(x)$ is a polynomial with (only) real coefficients, then:
$a+bi$ is a zero of $f(x) \Leftrightarrow a-bi$ is a zero of $f(x)$.

In our last Example in Part C, if we know that $i\sqrt{3}$ is a zero of $f(x)$, then we can conclude that $-i\sqrt{3}$ must also be a zero.

Technical Note: The theorem requires real coefficients. Observe that $x-i$ has i as a zero but not $-i$.

Note: There is a Conjugate Pairs Theorem for a quadratic polynomial $f(x)$ with (only) rational coefficients. Consider $f(x)=ax^2+bx+c$, where $a, b, c \in \mathbf{Q}$ and $a \neq 0$. As an example, $2+\sqrt{3}$ is a zero of such an $f(x) \Leftrightarrow 2-\sqrt{3}$ is.
The structure of the QF implies this.