PART E: THE FUNDAMENTAL THEOREM OF ALGEBRA (FTA)

The Fundamental Theorem of Algebra (FTA)

If $f(x)$ is a nonconstant n^{th}-degree polynomial in standard form with real coefficients, then it must have at least one complex (possibly real) zero.

Put Another Way: It must have exactly n complex zeros, where the zeros may be repeated based on their multiplicities.

Technical Note: The Fundamental Theorem of Arithmetic states that any integer greater than or equal to 2 is either prime or can be decomposed uniquely as a product of (possibly repeated) primes (or "prime powers"), up to a reordering of the factors. For example, 6 can only be decomposed in one way: $6 = 2 \cdot 3$. The decomposition $6 = 3 \cdot 2$ does not count as a different one.

Technical Note: The Fundamental Theorem of Calculus will allow you to evaluate definite integrals, which are used in finding areas, volumes, arc lengths, surface areas, and much more.

Historical Note: The FTA was first proven by the great Gauss. For more history, see p.193 of the textbook.

PART F: THE LINEAR FACTORIZATION THEOREM (LFT)

The Linear Factorization Theorem (LFT)

If $f(x)$ is a nonconstant polynomial in standard form with real coefficients, then it must have a factorization into linear factors of the form:

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$$
$$\left(a_n \in \mathbf{R};\ a_n \neq 0;\ \text{each } c_i \in \mathbf{C}\right)$$

Note: The zeros of $f(x)$ are then $c_1, c_2, \ldots, c_n$.

Note: There may be repetitions of a zero c_i, based on the multiplicity of c_i.

Note: a_n is the leading coefficient of $f(x)$.

Technical Note: The LFT is proven using the FTA and the Factor Theorem. See p.193 of the textbook.

Technical Note: This helps explain the Complex Conjugate Pairs Theorem in Notes 2.56.

Example

Let $f(x) = x^5 - 8x^4 + 16x^3$.

$$\begin{aligned} x^5 - 8x^4 + 16x^3 &= x^3\left(x^2 - 8x + 16\right) \\ &= x^3(x-4)^2 \end{aligned}$$

It may be said that $f(x)$ has 5 zeros: 0, 0, 0, 4, and 4.

$f(x)$ has only 2 distinct zeros: 0 and 4.
They are both repeated zeros:
The multiplicity of 0 is 3, and the multiplicity of 4 is 2.

You can think of x as $(x - 0)$.

Recall our Examples in Notes 2.52 and 2.53.

$$4x^3 - 5x^2 - 7x + 2 = (x-2)(4x-1)(x+1) \quad \leftarrow \text{Factored over } \mathbf{Z}$$
$$= 4(x-2)\left(x - \frac{1}{4}\right)(x+1) \quad \leftarrow \text{Factored over } \mathbf{Q}$$

The second factorization is in "LFT Form." The zeros can be immediately read off (watch out for signs, though).

PART G: FACTORING OVER R

"Factoring Over **R**" Theorem

Let $f(x)$ be a nonconstant polynomial in standard form with real coefficients. Its complete factorization over **R** (the reals) consists of:

1) Linear factors,
2) Quadratic factors that are irreducible over **R** (i.e., have no real zeros), or
3) Some product of the above, possibly including repeated factors, and
4) Maybe a nonzero constant factor.

One consequence: A 3^{rd}- or higher-degree polynomial $f(x)$ with real coefficients **must** be factorable (reducible) over the reals. Knowing **how** to factor such an $f(x)$ may pose a problem, however!

Note: We will need this theorem when we do Partial Fraction Decompositions in Section 2.7.

Technical Note: See the Proof on p.193 of the textbook. Consider the "LFT Form" of $f(x)$. If all the zeros of $f(x)$ are real, then it can be factored accordingly. If there exists an imaginary zero c_i, then its conjugate $\overline{c_i}$ must also be a zero by the Complex Conjugate Pairs Theorem, and the product $(x - c_i)(x - \overline{c_i})$ of their corresponding factors must have real coefficients (see p.193); this product would be a quadratic factor of $f(x)$ with real coefficients that is irreducible over **R** (because its zeros are not real). Keep pairing off complex conjugate pairs of imaginary zeros until the remaining factors have only real coefficients.

PART H: DESCARTES'S RULE OF SIGNS

Historical Note: In addition to the Cartesian plane, this is also named after René Descartes. See Notes P.27.

Assumptions and Preliminaries

Let $f(x)$ be a polynomial with real coefficients written in standard form:

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

$$\left(\text{each constant } a_i \in \mathbf{R};\ a_n \neq 0;\ a_0 \neq 0;\ n \in \mathbf{Z}^+\right)$$

Note: As with the Rational Zero Test, we require $a_0 \neq 0$. If $a_0 = 0$, factor out the GCF first. For example, $x^3 + x^2$ factors as $x^2(x+1)$, and we know that 0 is a real zero of multiplicity 2.

Variations in Sign

The number of variations in sign in $f(x)$ is given by the number of "sign flips" as the nonzero coefficients of $f(x)$ are read from left to right in the standard form.

Example

Find the number of variations in sign in $f(x) = 7x^6 - 2x^3 + 4x + 5$.

Solution

Because the leading coefficient is positive, we may want to clearly place a + sign in front of it:

$$f(x) = +7x^6 - 2x^3 + 4x + 5$$

There are **2** variations in sign. (Don't worry about "missing terms"; they have 0 coefficients.)

Parity

Two integers have the same parity $\Leftrightarrow$ They are both even or both odd.

Descartes's Rule of Signs

We want information about z^+ and z^-, where:

z^+ is the number of **positive** real zeros of $f(x)$, and
z^- is the number of **negative** real zeros of $f(x)$.

Let v^+ be the number of variations in sign in $f(x)$ (written in standard form).
Let v^- be the number of variations in sign in $f(-x)$ (written in standard form).

Then,

$0 \le z^+ \le v^+$, where z^+ has the same parity as v^+, and
$0 \le z^- \le v^-$, where z^- has the same parity as v^-.

Warning: A zero of multiplicity k is counted k times here.
For example, $f(x) = x^3 + 3x^2 + 3x + 1$, which factors as $(x+1)^3$, is said to have 3 real zeros: -1, -1, and -1.

Example

Based on Descartes's Rule of Signs, give the possible values of z^+ and z^- for $f(x) = 4x^3 - 5x^2 - 7x + 2$. (We've used this $f(x)$ in previous sections.)

Solution

Find possible values for z^+:

$$f(x) = +4x^3 - 5x^2 - 7x + 2$$

We see that $v^+ = 2$, an even integer.
Therefore, $0 \le z^+ \le 2$, where z^+ is also even.
The possible values for z^+ are then **0 and 2**.

Tip: Observe that we start with 2 and count down by twos; we stop before reaching negative numbers. This is similar to listing possible numbers of turning points for polynomial graphs in Section 2.2.

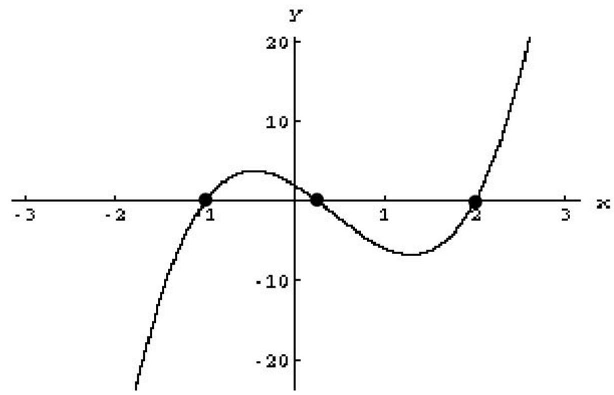
Find possible values for z^-:

$$\begin{aligned} f(-x) &= 4(-x)^3 - 5(-x)^2 - 7(-x) + 2 \\ &= -4x^3 - 5x^2 + 7x + 2 \end{aligned}$$

We see that $v^- = 1$, an odd integer.
Therefore, $0 \le z^- \le 1$, where z^- is also odd.
The **only** possible value for z^- is **1**.

(In other words, $f(x)$ must have exactly one negative real zero.)

Note: We earlier found that $f(x)$ had **2** positive real zeros, namely 2 and $\frac{1}{4}$, and **1** negative real zero, namely -1. See the graph below.



Note: Consider the form $f(x) = x^n \pm 1$ as a source of basic examples.

PART I: UPPER AND LOWER BOUND RULES FOR ZEROS

> a is a lower bound for the real zeros of f, and b is an upper bound for them $\Leftrightarrow$ **All** the real zeros of f lie in the interval $\left[a, b\right]$.

It is easier to demonstrate the Upper and Lower Bound Rules rather than to state them in general.

We require that the leading coefficient of $f\left(x\right)$, a_n, be positive, and that all the coefficients be real.

Example and Demonstration

Show that all the real zeros of $f\left(x\right) = 4x^3 - 5x^2 - 7x + 2$ must lie in the interval $\left[-1, 3\right]$.

Solution

Use Synthetic Division to divide $f\left(x\right)$ by $x - 3$:

3	4	−5	−7	2
		12	21	42
	4	7	14	(44)

Because $\mathbf{3 > 0}$, and all the entries in the last row are **nonnegative**, 3 is an **upper bound** for the real zeros of f.

Use Synthetic Division to divide $f(x)$ by $x-(-1)$:

-1	4	-5	-7	2
		-4	9	-2
	4	-9	2	0

Because $\mathbf{-1 < 0}$, and the entries in the last row **alternate between nonnegative and nonpositive entries**, -1 is a **lower bound** for the real zeros of f.

In fact, because we get a 0 remainder, -1 must be a zero of f.

Therefore, all the real zeros of $f(x)$ must lie in the interval $[-1, 3]$.

Note: These rules can be used (possibly in conjunction with Descartes's Rule of Signs and/or a graph) to shrink the list of candidates for zeros resulting from the Rational Zero Test. The information obtained from these rules can also help us use the Intermediate Value Theorem (see Notes 2.20-2.21 on Section 2.2) more effectively in attempting to locate where zeros may be.