# SECTION 2.6: RATIONAL FUNCTIONS

## PART A: ASSUMPTIONS

Assume $f(x)$ is rational and written in the form $f(x)=\dfrac{N(x)}{D(x)}$,

where $N(x)$ and $D(x)$ are polynomials, and $D(x)\neq 0$ (i.e., the zero polynomial).

Assume **for now** that $N(x)$ and $D(x)$ have no real zeros in common.

Note: The textbook essentially makes this last assumption when it assumes that $N(x)$ and $D(x)$ have no common factors (over **R**) aside from $\pm 1$, though, in Part E, we will consider what happens when we relax this assumption.

Warning: Even though $\dfrac{x^2+x}{x}=x+1 \quad (\forall x\neq 0)$, we do **not** consider the rational function [rule] $f(x)=\dfrac{x^2+x}{x}$ to be a polynomial function [rule].

## PART B: VERTICAL ASYMPTOTES (VAs)

An asymptote for a graph is a line that the graph approaches.

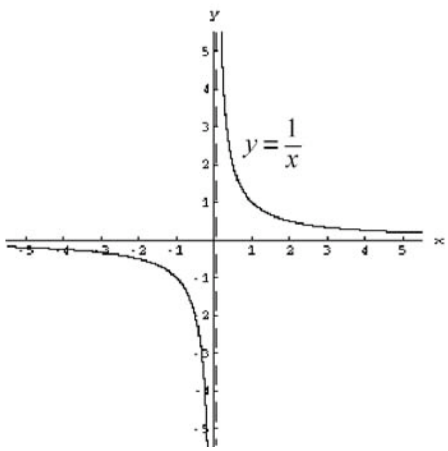
Example

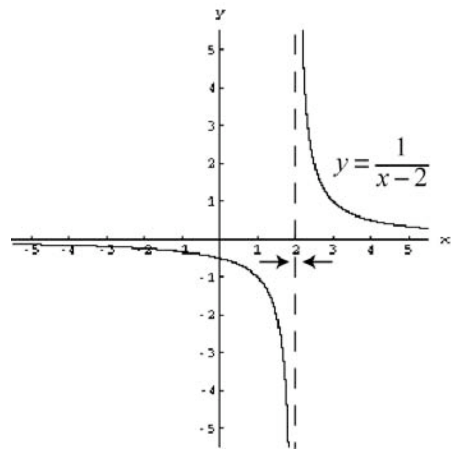
Let $f(x)=\dfrac{1}{x-2}$. Find any VAs for the graph of $f$.

Solution

Observe that 1 and $x-2$ have no real zeros in common.
$x-2=0 \iff x=2$, so the only VA has equation $\boldsymbol{x=2}$.

The graph of $y=\frac{1}{x}$ (on the left) is translated 2 units to the right to obtain the graph of $y=\frac{1}{x-2}$ (on the right). We typically use dashed lines to indicate asymptotes.




$x=2$ is a VA for the graph on the right, because:
(Actually, just one of the two statements below would be sufficient.)

$f(x)\to\infty$ as $x\to 2^{+}$
(i.e., as $x$ approaches 2 <u>from the right</u>, or from higher numbers), and

$f(x)\to-\infty$ as $x\to 2^{-}$
(i.e., as $x$ approaches 2 <u>from the left</u>, or from lesser numbers).

| | |
|---|---|
| | Under our Assumptions in Part A, the graph of $f(x)=\frac{N(x)}{D(x)}$ has a VA at $x=c$ $(c\in\mathbf{R})$ |
| $\Leftrightarrow$ | $c$ is a real zero of $D(x)$ |
| $\Leftrightarrow$ | $f(x)\to\infty$ or $-\infty$ as $x\to c^{+}$, and $f(x)\to\infty$ or $-\infty$ as $x\to c^{-}$. |

<u>Note</u>: When we study logarithmic functions in Chapter 3, we will see graphs that have "one-sided" VAs. Graphs of rational functions "shoot off" on both sides of any VAs.

## PART C: HORIZONTAL ASYMPTOTES (HAs)

The graph of $f(x)=\dfrac{N(x)}{D(x)}$ has a HA at $y=L$ $(L \in \mathbf{R})$

$\Leftrightarrow$ $f(x) \to L$ as $x \to \infty$ **and** as $x \to -\infty$.

Note: The graph of a **rational** function can have **at most one** HA.
The graph of a function that is **not rational** can have **at most two** HAs; $f(x)$ may approach different real values when $x \to \infty$ as opposed to when $x \to -\infty$.

Case 1

If $\deg(N) < \deg(D)$, then $f$ is a proper rational function, and the $x$-axis $(y=0)$ is the only HA of its graph.

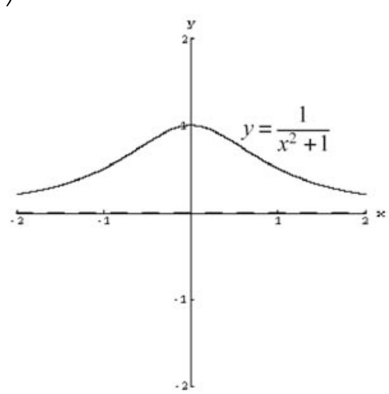
Example: $f(x)=\dfrac{1}{x}$. See the graph on Notes 2.67.

Example

Consider $f(x)=\dfrac{1}{x^2+1}$.

$\deg(N) < \deg(D)$, because $0<2$.
The $x$-axis is the only HA of the graph of $f$.

Observe that $f(x)>0$ $(\forall x \in \mathbf{R})$, $f$ is even, and its graph (below) has no VAs, because $D(x)=x^2+1$ has no real zeros.



Case 2

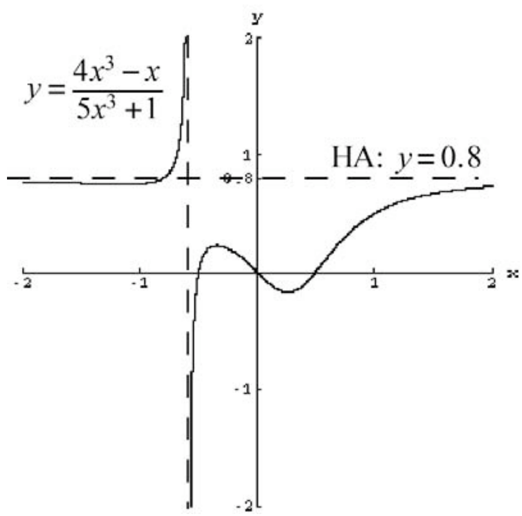
If $\deg(N) = \deg(D)$, then $y = L$ is the only HA of the graph of $f$,
where $L = \dfrac{\text{the leading coefficient of } N(x)}{\text{the leading coefficient of } D(x)}$.

Example

Consider $f(x) = \dfrac{4x^3 - x}{5x^3 + 1}$.

$\deg(N) = \deg(D)$, because $3 = 3$.

$y = \dfrac{4}{5}$ (or 0.8) is the only HA of the graph of $f$ (below).




Warning: Observe from the graph above that it is possible for the graph of $y = f(x)$ to cross a HA. However, its graph can never cross a VA.

Note: The idea is that, in the "long run," the "Zoom Out" Dominance Property for polynomials applies to the numerator and the denominator:

$$f(x) = \frac{4x^3 - x}{5x^3 + 1} \approx \frac{4x^3}{5x^3} = \frac{4}{5} \text{ if } x \text{ is "extreme"}$$

<u>Case 3</u>

If $\deg(N) > \deg(D)$, then the graph of $f$ has no HAs.

The "Zoom Out" Property described in Part D will tell us about the "long-run" behavior of such graphs.

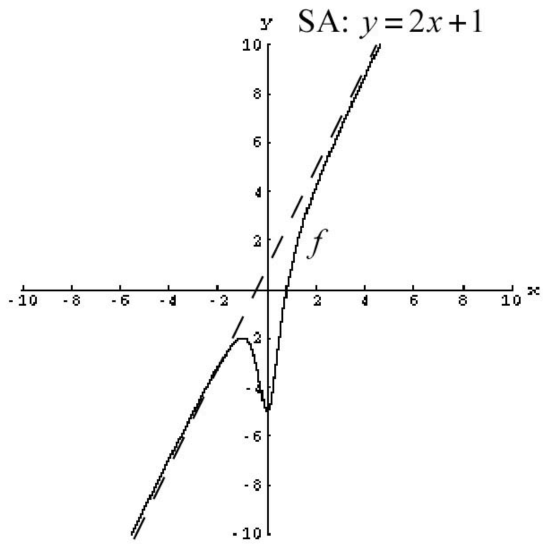
## PART D: THE "ZOOM OUT" PROPERTY FOR RATIONAL FUNCTIONS; SLANT ASYMPTOTES

<u>Example</u> (from Section 2.3, Part A)

We have used long division to express $f(x) = \dfrac{-5 + 3x^2 + 6x^3}{1 + 3x^2}$ as:

$$f(x) = \underbrace{2x + 1}_{\substack{\text{polynomial} \\ \text{part, } p(x)}} \quad \underbrace{- \frac{2x + 6}{3x^2 + 1}}_{\substack{\text{proper rational} \\ \text{part, } r(x)}}$$

This makes the corresponding graph (below) much easier to analyze.



Case 1 from Part C tells us that $r(x) \to 0$ as $x \to \infty$ **and** as $x \to -\infty$.
(The proper rational part "decays" in the long run.)
Therefore, the graph of $f(x)$ approaches the graph of $p(x)$ as $x \to \infty$ **and** as $x \to -\infty$.

Here, the graph of $f(x)$ approaches the graph of $p(x) = 2x + 1$.
Because $p(x)$ is linear, we call $y = 2x + 1$ a slant asymptote (SA) or an oblique asymptote for the graph of $f$.

Note: The graph of $f$ has a SA $\Leftrightarrow$ $\deg(N) = \deg(D) + 1$.

Note: The graph of $f$ has no VAs, because $D(x) = 3x^2 + 1$ has no real zeros.

## PART E : WHAT IF $N(x)$ AND $D(x)$ HAVE REAL ZEROS IN COMMON?

The graph of $f$ may have a VA or a hole at such a common real zero.

In Calculus: The "limit form" $\frac{0}{0}$ is important in Calculus. The limit definitions of the derivative described in Notes 1.24-1.25 and 1.57-1.58 involve this form.

Example

Graph $f(x) = \dfrac{x^2 - 9}{x - 3}$.

Solution

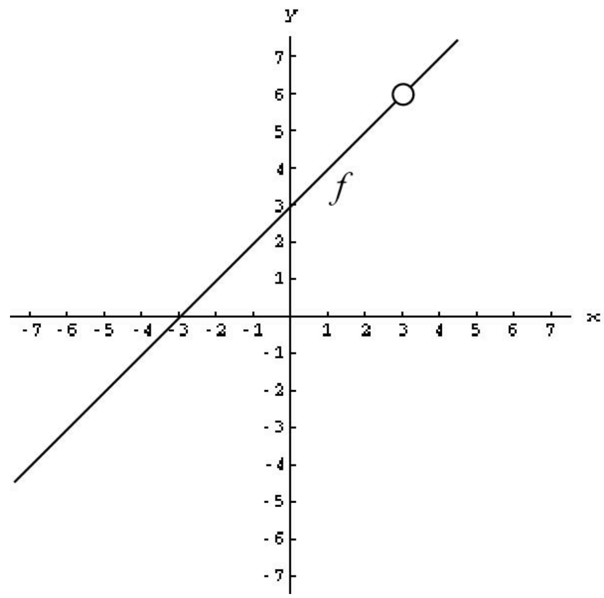
Observe that 3 is a real zero of $D(x) = x - 3$, so 3 is excluded from the domain of $f$, and 3 is also a real zero of $N(x) = x^2 - 9$. By the Factor Theorem, $(x - 3)$ must be a factor of $N(x)$.

$$
\begin{aligned}
f(x) &= \frac{x^2 - 9}{x - 3} \\
&= \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} \\
&= x + 3 \quad (x \neq 3)
\end{aligned}
$$

We include $(x \neq 3)$ as part of the final expression, because it is not apparent from the expression $x + 3$ that 3 is excluded from the domain of $f$.

The graph of $f$ is essentially the line $y = x + 3$, except that the point $(3, 6)$ is deleted from the graph.

In Calculus: We say that the resulting “hole” reflects the fact that $f$ has a removable discontinuity at $x = 3$.



Example

Graph $f(x)=\dfrac{x-2}{x^2-4x+4}$.

Solution

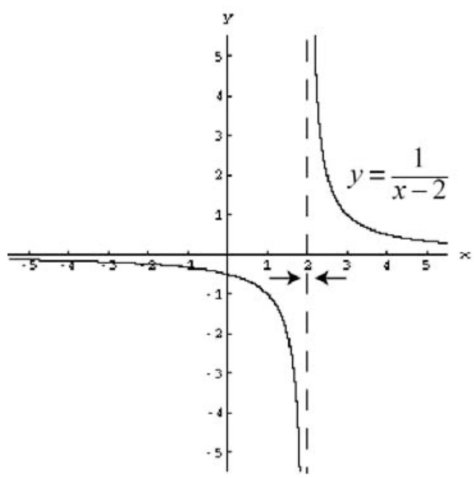
$$f(x)=\frac{x-2}{x^2-4x+4}$$
$$=\frac{x-2}{(x-2)^2}$$

Observe that 2 is the only real number excluded from $\text{Dom}(f)$.

$$=\frac{1}{x-2}$$

It is apparent from the final expression, $\dfrac{1}{x-2}$, that 2 is excluded from the domain of $f$, so we need not write $(x\neq 2)$.

The graph of $f$ is the same as the second graph from Part B: Notes 2.67:




We have a VA and not a hole at $x=2$, because not all of the $(x-2)$ factors in the denominator were canceled (divided) out in the simplification process. In our previous Example, all of the $(x-3)$ factors were canceled (divided) out in the denominator.

**PART F: COMMENTS ON GRAPHING RATIONAL FUNCTIONS (BONUS TOPIC)**

Our prior observations, in conjunction with the "Zoom Out" Dominance Property for polynomials, tell us that, in the "long run," graphs of rational functions look like lines, bowls, or snakes.

See Notes 2.22-2.24 on Section 2.2, Part H.
When considering the graph of a rational function $f$, we make the following modifications to those Notes:

Determine the domain of $f$. Find any VAs, HAs (see #4 below), and holes.

Remember that graphs of rational functions have no cusps or sharp corners (such as for $|x|$).

1) Find the $y$-intercept, if any.

   If 0 is not in the domain of $f$, then there is no $y$-intercept.

2) Find the $x$-intercept(s), if any.

   When determining the real zeros of $f$, make sure that your zeros are, in fact, in the domain of $f$. Look for real zeros of $N(x)$ that are not zeros of $D(x)$.

3) Exploit symmetry, if possible.

   Is $f$ even? Odd?

4) Determine the "long-run" behavior of the graph as $x \to \infty$ and as $x \to -\infty$.

   Use Part C to find any HAs. Use Part D and maybe basic algebra or Long or Synthetic Division to find SAs or other "long-run polynomial behaviors"; remember the "Zoom Out" Properties for both rational and polynomial functions.

5) Find where $f(x) > 0$ and where $f(x) < 0$.

Let's modify our comments on the Test Interval (or "Window") Method from Section 2.2:

**A rational function can only change sign …**

- **… at its zeros, or**
- **… where it is undefined (i.e., where there is a hole or a VA).**

Note: A rational function can only change sign at a hole if it lies on the *x*-axis.

6) Maybe do some point-plotting (for a more accurate graph).

For many more examples, see the figures in the textbook.