SECTION 2.7: NONLINEAR INEQUALITIES

We solved linear inequalities to find domains, and we discussed intervals in Section 1.4: Notes 1.24 to 1.30.

In this section, we will solve nonlinear inequalities to find domains.

Example 1

Let $f(x) = \sqrt{x^2 - 9}$.

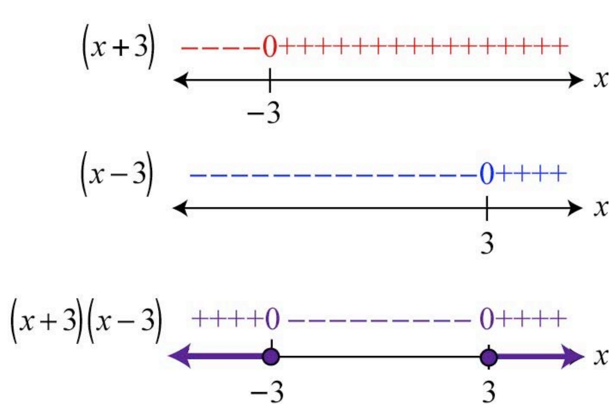
We get real outputs $\Leftrightarrow$ $x^2 - 9 \geq 0$. There are different ways to solve this inequality; its solution set is the domain of f.

Method 1: Sign Chart Method;
we are solving $x^2 - 9 \geq 0$

The key idea here is that we'd rather perform a sign analysis on products of factors as opposed to sums of terms. (For example, the product of a positive real number and a negative real number is guaranteed to be negative; however, there is no such guarantee regarding their sum.) Factoring can be a key tool.

$$x^2 - 9 \geq 0$$
$$(x+3)(x-3) \geq 0$$

We need to determine where each of the factors on the left side is negative, 0, and positive in value. We ultimately want to know where their product is 0 or positive.



The domain of f is: $(-\infty, -3] \cup [3, \infty)$.

Method 2: Parabola Method (for Quadratic Inequalities);
we are solving $x^2 - 9 \geq 0$

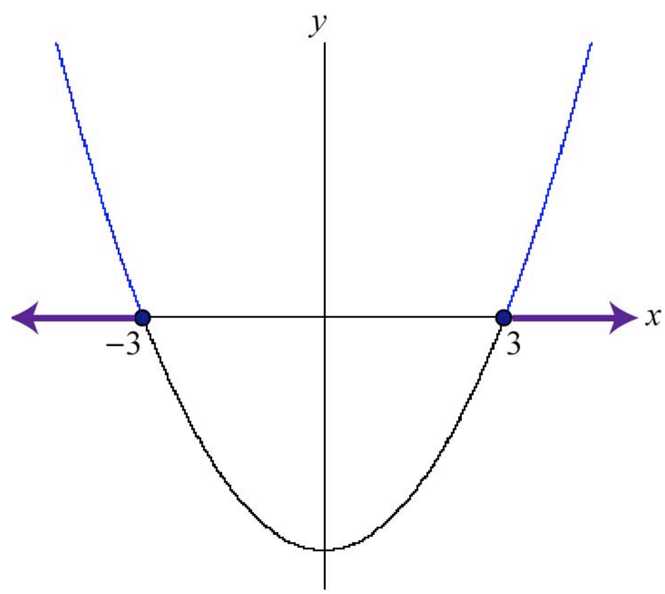
The real zeros of $x^2 - 9$ are the x-intercepts of the corresponding parabola:

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

Warning: Here, the $\pm$ symbol means "take **both** the $+3$ and the -3."
See Warning 1 in Section 1.5: Notes 1.46.

The leading coefficient of $x^2 - 9$ is positive, so the parabola opens up.

This is enough information for us to sketch the parabola to our satisfaction.



Given an input x, the y-coordinate of the corresponding point gives the output (or function value).

Because of the "≥ 0" in our inequality, we need the values of x, if any, that correspond to the parts of the parabola that lie above or on the x-axis.

Again, the domain of f is: $(-\infty, -3] \cup [3, \infty)$.

See also the bottom of p.198.

Method 3: Test Value or Test Interval Method;
we are solving $x^2 - 9 \geq 0$

This method presented in Larson, Section 2.7 may be the most straightforward one for rational inequalities in general. We discussed these ideas for continuous, particularly polynomial, functions in Section 2.2: Notes 2.23, and we extended them to rational functions in Section 2.6: Notes 2.76:

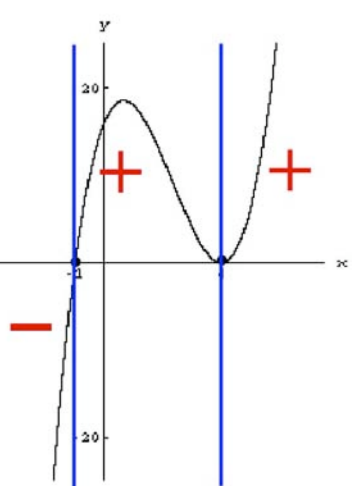
A rational function can only change sign …   • … at its zeros, or   • … where it is undefined (i.e., where the graph has a hole or a VA).

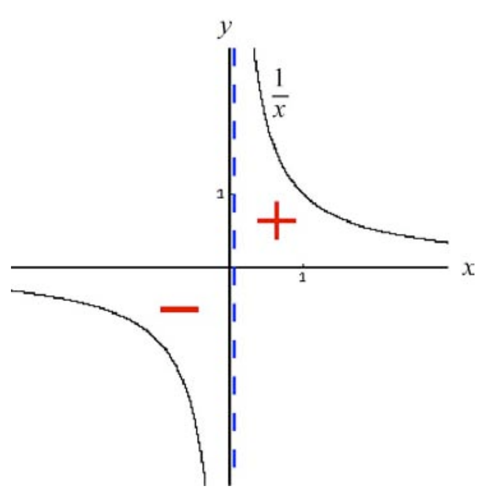
Warning: These places are called critical numbers in Larson, but this term has a different meaning in Calculus. In Calculus I, they are numbers in the domain of a function where the function's derivative (see Notes 1.63-1.66) is 0 or undefined; see Notes 1.49 to get a hint as to why we care ….

Warning: The function may or may not change sign at these places.

For example, look at the graphs of $y = x^3 - 7x^2 + 8x + 16$ and of $y = \frac{1}{x}$ below.

Note: The blue window separators marking the zeros in the first graph are not asymptotes, but the blue dashed line marking where $\frac{1}{x}$ is undefined in the second graph is.





Remember, we are solving $x^2 - 9 \ge 0$. Now, $x^2 - 9$ is a polynomial in x, so it is never undefined, and its zeros are 3 and -3, as we have seen in Method 2.

We use 3 and -3 as breakpoints (or fence posts) along the real number line. They break up the number line into three open test intervals, excluding 3 and -3, themselves. We test an x-value in each of the three intervals, and we at least determine the **sign** of $x^2 - 9$ at that test x-value; the sign there must be the common sign throughout the entire interval (see the box on the previous page).

		-3		**3**	
Test x-values	-4		0		4
Value of $x^2 - 9$	$(-4)^2 - 9 = 7$	0	$(0)^2 - 9 = -9$	0	$(4)^2 - 9 = 7$
Sign of $x^2 - 9$	+	0	–	0	+

We want the values of x for which $x^2 - 9$ is either "+" or "0."

Again, the domain of f is: $(-\infty, -3] \cup [3, \infty)$.

Note: Instead of testing 4 and -4, you may want to test extreme values such as 100 and -100. The corresponding signs may be easier to figure out. Bear in mind that you do not need to find the corresponding numerical values of $x^2 - 9$; only signs matter. In fact, the following variation is often an improvement ….

Method 4: Test Value or Test Interval Method using Factored Forms;
we are solving $x^2 - 9 \ge 0$

The Study Tip on p.199 of Larson suggests the sign analysis of factored forms, which can make our work for Method 3 more efficient, especially for more complicated inequalities. This method is a hybrid of Methods 1 and 3. Remember that $x^2 - 9 = (x+3)(x-3)$. We revise the chart from Method 3 as follows:

		-3		**3**	
Test x-values	-4		0		4
Signs: $(x+3)(x-3)$	$(-)(-)$	0	$(+)(-)$	0	$(+)(+)$
Sign of Product, $x^2 - 9$	+	0	–	0	+

Example 2

Let $f(x) = \frac{1}{\sqrt{x^2 - 9}}$.

This is similar to Example 1, except that we get real outputs $\Leftrightarrow$ $x^2 - 9 > 0$.
Note that we exclude 0, itself, here.

In our graphs for the first two methods in Example 1, we replace filled-in circles with hollow ones.

In our charts for the last two methods, the "0"s in the bottom lines are no longer in red.

We also replace brackets with parentheses in our answer, because we must exclude the endpoints -3 and 3 from the domain.

The domain of f is: $(-\infty, -3) \cup (3, \infty)$.

Example 3

Let $f(x) = \sqrt[3]{x^2 - 9}$.

The domain of f is **R**, because:

• $x^2 - 9$ is a polynomial with unrestricted domain, and

• (Warning!) The taking of **odd** roots (such as cube roots) does **not** impose any new restrictions on the domain. Remember that the cube root of a negative real number is a negative real number. This is different from **even** roots (such as square roots); we do not permit even roots of negative numbers when we find a domain.

For more on domains of radical functions, see Notes P.19.

Warning 1: When dealing with inequalities, if you multiply or divide both sides by a negative quantity, you must **reverse** the direction of the inequality symbol.
For example, $-x < -2 \quad \Leftrightarrow \quad x > 2$. You must also reverse the direction if you switch the left side and the right side. For example, $a < b \quad \Leftrightarrow \quad b > a$.

Warning 2: Do not multiply or divide both sides of an inequality by a variable expression, unless you take the time to consider cases when the expression is positive, zero, and negative in value.

Warning 3: When solving a nonlinear inequality such as $x^2 - 9 \geq 0$, make sure 0 is isolated on one side if you are going to use one of the methods from Example 1. This is because sign analyses are based on comparisons with 0.

Example (Warnings 2 and 3)

When solving the inequality $x^2 > x$, do not divide both sides by x.
Instead, isolate 0 on one side by subtracting x from both sides:

$$x^2 > x$$
$$x^2 - x > 0$$

Then, use one of the methods given in Example 1.
The solution set turns out to be: $(-\infty, 0) \cup (1, \infty)$.
These are the numbers whose squares are greater than themselves.

Warning 4: Let's say we have $f(x) = \sqrt{9 - x^2}$, and we want to use the Sign Chart Method (Method 1). Remember that $9 - x^2$ factors as $(3 + x)(3 - x)$, which is the **opposite** of $(x + 3)(x - 3)$ because of the "Switch Rule for Subtraction."

Example 3 on p.200 in Larson deals with "Unusual Solution Sets."