CHAPTER 3: EXPONENTIAL AND LOG FUNCTIONS

SECTION 3.1: EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

PART A: THE LEGEND OF THE CHESSBOARD

The original story takes place in the Middle Ages and involves grains of wheat. Instead, we shall transport ourselves to the distant realm of Seattle, where a smart programmer is haggling with King Bill. The programmer agrees to work for King Bill for 63 days, starting tomorrow. After seeing a large chessboard engraved into King Bill's floor, the programmer comes up with a scheme for his salary. For now, the programmer tells King Bill to place a check for $1 on "Square 0" on his chessboard. With each new workday, King Bill is to place twice as much money on the corresponding square as the day before. The chortling King Bill, who has forgotten all of his math, agrees. What will happen?

The chessboard:

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
8	**9**	**10**	**11**	**12**	**13**	**14**	**15**
16	**17**	**18**	**19**	**20**	**21**	**22**	**23**
24	**25**	**26**	**27**	**28**	**29**	**30**	**31**
32	**33**	**34**	**35**	**36**	**37**	**38**	**39**
40	**41**	**42**	**43**	**44**	**45**	**46**	**47**
48	**49**	**50**	**51**	**52**	**53**	**54**	**55**
56	**57**	**58**	**59**	**60**	**61**	**62**	**63**

The amount of money (in dollars) placed on square x is given by $f(x) = 2^x$.

Here are some sample values:

Square x	$f(x)$ (in \$)
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
20	Over 1 million (i.e., 10^6)
30	Over 1 billion (i.e., 10^9)
40	Over 1 trillion (i.e., 10^{12})
63	Over 9 quintillion (i.e., 9×10^{18})

The amount of money on Square $(x+10)$ will be over 1000 times the amount of money on Square x $(0 \leq x \leq 53)$, because the multiplier is $2^{10} = 1024$.

Challenge: How much money should be on the entirety of the chessboard after Day 63? Hint: Experiment with the first few days. We will see a relevant formula in Chapter 9, when we get to finite geometric series.

Remember that our national debt is "only" in the trillions.

No wonder we associate this kind of exponential growth with "rapid growth" in our language!

PART B: BASIC EXPONENTIAL GRAPHS

We call b a "nice base" if $b > 0$ and $b \neq 1$.

Basic exponential functions have the form $f(x) = b^x$, where b is nice.

Example

Graph $f(x) = 2^x$, our "payment" function from Part A.

Solution

The table on the previous page gives some sample points (x, y).

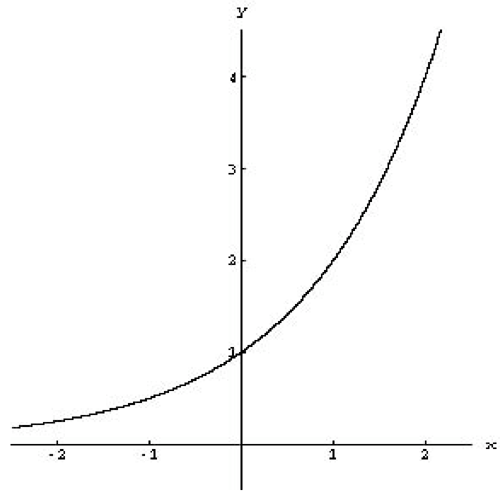
However, the domain of 2^x is assumed to be **R**, not just the nonnegative integers.

Technical Note: Let's look at $2^{3/4}$, for example. We may interpret $2^{3/4}$ as $\sqrt[4]{2^3}$, or $\sqrt[4]{8}$. It is the real number whose fourth power equals 8. The idea makes sense, although the number, which is irrational, may be time-consuming to approximate by "trial-and-error" on a calculator. You will encounter helpful methods and tools such as Newton's Method in Calculus I: Math 150 and series in Calculus II: Math 151 at Mesa. The explanation for values of 2^x for **irrational** values of x is actually a calculus idea, in and of itself! See p.198 of the textbook.

What about when $x < 0$? Observe the pattern:

$\boldsymbol{x}$	$f(x) = 2^x$
3	8
2	4
1	2
0	1
−1	$2^{-1} = \frac{1}{2}$
−2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
−3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

Here is the graph of $f(x) = 2^x$:



In general, if $b > 1$ …

> The graph of $f(x) = b^x$, where $b > 1$, will resemble the "J" graph above.
> Think: <u>Exponential growth</u>.

For $f(x) = b^x$, where b is **any** nice base:

- The domain is **R**.
- The range is $(0, \infty)$.
- The x-axis is a horizontal asymptote for the graph.
- The y-intercept is 1, because $b^0 = 1$.

The various transformations from Section 1.6 apply here, as well.

What about if $0 < b < 1$?

Example

Graph $g(x) = \left(\frac{1}{2}\right)^x$ by first considering $f(x) = 2^x$.

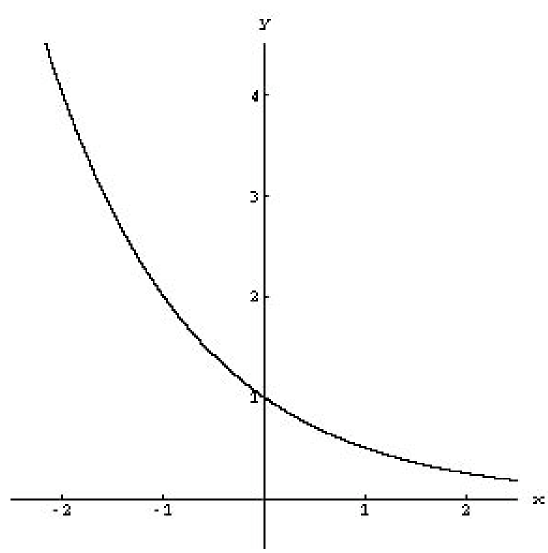
Solution

Observe:

$$\begin{aligned} f(-x) &= 2^{-x} \\ &= \left(2^{-1}\right)^x \\ &= \left(\frac{1}{2}\right)^x \\ &= g(x) \end{aligned}$$

We reflect the old graph about the *y*-axis to obtain the new graph.

Here is the graph of $g(x) = \left(\frac{1}{2}\right)^x$:



In general, if $0 < b < 1$ …

The graph of $f(x) = b^x$, where $0 < b < 1$, will resemble the "curvy L" graph above. Think: Exponential decay.

What happens if *b* is **not** a nice base? (Optional discussion)

What happens if $b = 1$?

$f(x) = 1^x = 1$ is a constant function, not an exponential function.

What happens if $b = 0$?

Observe that $f(x) = 0^x = 0$, if $x \neq 0$.

We sometimes have to conveniently define 0^0 ourselves, depending on our problem. Observe that 0^2, for example, is 0, yet 2^0 is 1. What would 0^0 be?

What happens if $b < 0$?

We have a real problem here. Literally. Think about the fact that $(-2)^2 = 4$, a positive real number, while $(-2)^3 = -8$, a negative real number. Meanwhile, $(-2)^{5/2} = \left(\sqrt{-2}\right)^5$ is not even a real number.

PART C: *e*

$e \approx 2.718$
Like π, it is an irrational number. They're pretty close in value, too!
There are different ways of defining *e*. Here's a limit definition for *e*:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Technical Note: In Math 151: Calculus II at Mesa, we say that we are dealing here with the limit form 1^∞, because the base of $\left(1 + \frac{1}{n}\right)^n$ is approaching 1, while the exponent is approaching ∞. It's as though *e* is a "long-term compromise" between these two countervailing forces. The form 1^∞ is called an indeterminate limit form, because it is not immediately clear what the limit should be in such cases, if there even is one. Different expressions with this limit form may have different limits. Other examples of indeterminate limit forms: 0^0, $0 \cdot \infty$, $\frac{0}{0}$, and $\frac{\infty}{\infty}$.

$f(x) = e^x$ gives us the natural exponential function, and *e* is referred to as the natural base. The graph of $f(x) = e^x$ resembles the J-graph for 2^x.

Technical Note: This function has many nice properties. For example, its derivative function is itself. That is not true of, say, the 2^x function.

PART D: EXPONENTIAL MODELS

Many applications employ the model $f(x) = a \cdot b^x$, where $a > 0$ and b is a nice base.

If $b > 1$ ("J" graphs), we obtain exponential growth models used in such applications as population growth (the Malthusian model) and compound interest, as we will see in Parts E and F.

> Historical Note: Thomas Malthus (1766-1834) was a famed economist who believed that populations would grow exponentially, but that food supplies would only grow linearly. His bleak views and proposed social remedies led people to call economics the "dismal science." (Microsoft® Encarta® Encyclopedia)

If $0 < b < 1$ ("curvy L" graphs), we obtain exponential decay models. For example, radioactive decay models are used in such applications as carbon-14 dating of ancient objects. If $b = \frac{1}{2}$, we deal with "half-life" models. See p.205.

PART E: COMPOUND INTEREST

Consider a banking account with compound interest where:

P = principal deposited (in dollars)

r = annual interest rate (as a decimal)

n = number of compoundings (i.e., number of times interest is paid) per year

t = time elapsed (in years) since the deposit

(P, r, n, and t must always be positive in value.)

Then, after t years, the account has:

$$f(t) = P\left(1+\frac{r}{n}\right)^{nt} \text{ dollars}$$

I will give you this formula on exams, if you need it.

This assumes that there are no withdrawals or deposits after the principal is deposited. We also assume that r stays constant for the time being. This may not be realistic!

Observe that the formula takes on a basic exponential form: $f(t) = a \cdot b^t$, where t is the independent variable, $a = P$, and the nice base $b = \left(1+\frac{r}{n}\right)^n > 1$.

Technical Note: We assume that t is always an integer, or at least that t represents a time at which interest is being compounded. Otherwise, if we allow t to represent any positive real number, we need to set up something that resembles a piecewise-defined function with a step graph.

Note: Simple interest is always applied to the principal only. Compound interest is applied to the combined total of the principal and the earned interest to date. The classic simple interest model is given by the formula $f(t) = P + Prt$, which is **linear** in t. Our compound interest models are **exponential** in t.

Note: We need the "1" term in the base. Otherwise, you are given interest, but then the rest is taken away!

Example

We initially deposit \$10,000 in an account that earns 6% annual interest compounded monthly. How much money will be in the account after 5 years, if no withdrawals or deposits are made in the meantime?

Solution

We have:

$$P = 10{,}000\ (\$)$$
$$r = 0.06$$
$$n = 12\ (\text{because there are 12 months in a year})$$
$$t = 5\ (\text{years})$$

Then,

$$f(t) = P\left(1+\frac{r}{n}\right)^{nt}$$
$$f(5) = 10{,}000\left(1+\frac{0.06}{12}\right)^{(12)(5)} \quad (\text{See Warning below!})$$
$$= 10{,}000\left(1+\frac{0.06}{12}\right)^{60} \quad (\text{We have 60 compoundings in 5 years.})$$
$$\approx \mathbf{\$13{,}488.50}$$

Warning: You should simplify the exponent immediately. Otherwise, you must use grouping symbols around the exponent when you use your calculator. For example, if we want to compute $2^{(3)(4)}$, it is **incorrect** to input 2 (exponent) 3 (times) 4 (equals) on your calculator. Because of the order of operations, that would give us $2^3 \cdot 4$, which is not correct. This is a **very** common type of error made by students!

Note: Observe that $f(0) = P$, the initial amount.

PART F: CONTINUOUS COMPOUND INTEREST

What happens as we let $n \to \infty$ in our compound interest formula? We do **not** earn infinitely many dollars in finite time. We are now dealing with continuous compound interest, in which case our account is always growing "continuously" (i.e., at each moment) over time.

Consider a banking account with continuous compound interest where (P, r, and t are defined as before):

$P =$ principal deposited (in dollars)

$r =$ annual interest rate (as a decimal)

$t =$ time elapsed (in years) since the deposit

(P, r, and t must always be positive in value.)

Then, after t years, the account has:

$$A = Pe^{rt} \quad \text{dollars}$$

Know this formula for exams.

Proof / Derivation (Optional)

Take the compound interest formula $f(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

Let $k = \frac{n}{r} \Rightarrow n = kr$. Observe: $k \to \infty \Leftrightarrow n \to \infty$.

$$\begin{aligned} &\lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} \\ &= \lim_{k\to\infty} P\left(1 + \frac{r}{kr}\right)^{krt} \\ &= \lim_{k\to\infty} P\Bigg[\underbrace{\left(1 + \frac{1}{k}\right)^{k}}_{\to e}\Bigg]^{rt} \qquad \text{(See the limit definition of } e \text{ in Part C.)} \\ &= Pe^{rt} \end{aligned}$$

Example (similar to our previous one)

We initially deposit $10,000 in an account that earns 6% annual interest compounded continuously. How much money will be in the account after 5 years, if no withdrawals or deposits are made in the meantime?

Solution

We have:

$$P = 10{,}000\,(\$)$$
$$r = 0.06$$
$$t = 5\,(\text{years})$$

Then,

$$\begin{aligned} A &= Pe^{rt} \\ &= 10{,}000e^{(0.06)(5)} \\ &= 10{,}000e^{0.3} \quad \left(\text{Warning: Simplify the exponent now!}\right) \\ &\approx \mathbf{\$13,498.59} \end{aligned}$$

Compare this to the **$13,488.50** obtained from **monthly** compounded interest.