SECTION 3.2: LOGARITHMIC (LOG) FUNCTIONS AND THEIR GRAPHS

PART A: LOGS ARE EXPONENTS

Example

Evaluate: $\log_3 9$

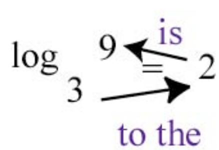
Solution

The question we ask is: "3 to what exponent gives us 9?"

$$\underbrace{\log_3 9 = \boxed{2}}_{\text{logarithmic form}} \text{, because } \underbrace{3^{\boxed{2}} = 9}_{\text{exponential form}}$$

We say: "Log base 3 of 9 is 2."

Think "Zig-zag":



Answer: 2.

More Examples

Log Form	Exponential Form
$\log_5 \left(\frac{1}{5} \right) = -1$	$5^{-1} = \frac{1}{5}$
$\log_9 3 = \frac{1}{2}$	$\underbrace{9^{1/2}}_{=\sqrt{9}} = 3$
$\log_{10} 10^7 = 7$	$10^7 = 10^7$

PART B: COMMON LOGS

$f(x) = \log_{10} x$ gives the common log function.
It is also written as simply: $f(x) = \log x$
(A missing log base is implied to be 10.)

Your calculator should have the LOG button.

Common logs are used in the Richter scale for measuring earthquakes and the pH scale for measuring acidity. Bear in mind that an earthquake measuring a "7" on the Richter scale is 10 times as powerful as one measuring a "6" and 100 times as powerful as one measuring a "5." Negative Richter numbers are also possible.

PART C: NATURAL LOGS

$f(x) = \log_e x$ gives the natural log function.
It is almost always written as: $f(x) = \ln x$

Your calculator should have the LN button.

In Calculus: This function is very useful, especially because its derivative is $\frac{1}{x}$.

Example: $\ln e^5 = \log_e e^5 = 5$

PART D: BASIC LOG PROPERTIES

Let b be any nice base.

Property	Because…	Special case $(b = e) \Rightarrow$	Memorize!
$\log_b 1 = 0$	$b^0 = 1$	$\log_e 1 = 0$	$\ln 1 = 0$
$\log_b b = 1$	$b^1 = b$	$\log_e e = 1$	$\ln e = 1$
$\log_b b^x = x$	$b^x = b^x$	$\log_e e^x = x$	$\ln e^x = x$
$b^{\log_b x} = x$ $(\text{if } x > 0)$	$\log_b x$ is the exponent that "takes us from b to x"	$e^{\log_e x} = x$	$e^{\ln x} = x$

We need the restriction $(x > 0)$ for the last property, because:

The log of a nonpositive number is not real.
In this class, we only take logs of **positive** real numbers.

Example: $\log_2(-1)$ is not real, because, if we set up $\log_2(-1) = \square$,
we see that $2^{\square} = -1$ has no real solution for $\square$.
This is because the range of the 2^x function is $(0, \infty)$, which excludes -1.

Example: Similarly, $\log_2 0$ is not real (in fact, it is undefined), because $2^{\square} = 0$ has no real solution.

Technical Note: In a course on complex variables, you will see that it is possible to take the log of a negative number (but not 0) in that setting.

Technical Note: We didn't need the restriction $(x > 0)$ for the other properties, because b and therefore b^x are presumed to be positive in value, anyway.

The last two properties are called inverse properties, because they imply that …

PART E: $f(x) = b^x$ AND $f^{-1}(x) = \log_b x$ REPRESENT INVERSE FUNCTIONS

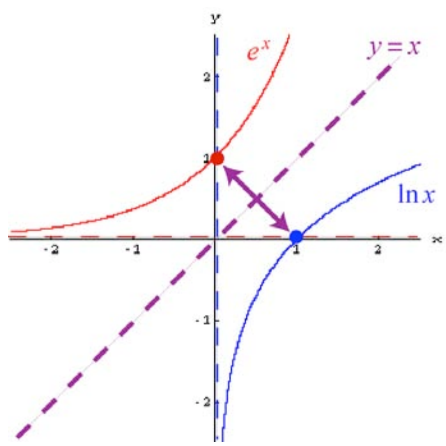
Let b be any nice base. The above exponential and log functions "undo" each other in that their composition in either order yields the identity function. (If the initial input is x, then the final output is x, at least if $x > 0$. See the last two properties in Part D.)

Example

$f(x) = e^x$ and $f^{-1}(x) = \log_e x$, or $\ln x$ represent a pair of inverse functions.

We know that the graph of $f(x) = e^x$ is a "J graph" similar to the one for 2^x. Observe that it passes the Horizontal Line Test (HLT), so f is one-to-one and therefore invertible.

We reflect this graph about the line $y = x$ to obtain the graph of the inverse function $f^{-1}(x) = \log_e x$, or $\ln x$.



Observe that the domain of one function is the range of the other, and vice-versa.

	Domain (x)	Range (y)	Asymptote for graph
e^x	**R**	$(0, \infty)$, the positive reals	x-axis
$\ln x$	$(0, \infty)$, the positive reals	**R**	y-axis

The graph for e^x has a y-intercept at $(0, 1)$, which reflects the fact that $e^0 = 1$.

The graph for $\ln x$ has an x-intercept at $(1, 0)$, which reflects the fact that $\ln 1 = 0$.

PART F: DOMAINS OF LOG FUNCTIONS

Example

Write the domain of $f(x) = \ln(x-3)$ in interval form.

Solution

$$\ln(x-3) \text{ is real} \quad \Leftrightarrow \quad x-3>0$$
$$\Leftrightarrow \quad x>3$$

The domain is: $(3, \infty)$

The graph of f is simply the graph for $\ln x$ shifted 3 units to the right: