SECTION 3.3: (MORE) PROPERTIES OF LOGS

PART A: READING LOG EXPRESSIONS

We will use grouping symbols as a means of clarifying the order of operations in expressions.

Often, grouping symbols are omitted when they could have helped.
How do we read log expressions in those cases?

We use ln for convenience, but any log function with any nice base is dealt with similarly.

Example: Read $\ln 3x + 7$ as: $\ln(3x) + 7$

Example: Read $\ln 3x^4$ as: $\ln(3x^4)$

Example: Read $\ln x^4$ as: $\ln(x^4)$, **not** as $(\ln x)^4$

Generally speaking, absent any grouping symbols, if we see "ln" or "log" followed by a product, quotient, power, or mixture of the above, the "ln" applies to the whole expression that follows. However, + and $-$ signs that introduce new terms tend to terminate the ln expression.

PART B: LOG PROPERTIES BASED ON LAWS OF EXPONENTS

Remember, logs are exponents. The laws for exponents imply laws for logs.

We will state these properties using ln, though they apply to any log function with a nice base.

If we use the rules from left-to-right, we are "expanding" the expression.
If we use the rules from right-to-left, we are "condensing" the expression.

Assume $A > 0$ and $B > 0$.
A and *B* may represent constants or variable expressions.

Product Rule

$$\ln(AB) = \ln A + \ln B$$

Think: The log of a **product** equals the **sum** of the logs.
(We go one step down in the order of operations if we read left-to-right.)

Related Exponent Law: When multiplying powers of *e*, the exponent on the product equals the sum of the exponents (of the factors): $e^A e^B = e^{A+B}$

Quotient Rule

$$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$$

Think: The log of a **quotient** equals the **difference** of the logs.

Related Exponent Law: When dividing powers of *e*, the exponent on the quotient equals the difference of the exponents: $\frac{e^A}{e^B} = e^{A-B}$

Power Rule

$$\ln A^p = p \ln A$$

Think: Smackdown Rule (from left-to-right);
Basketball Rule (from right-to-left)

Related Exponent Law: When raising a power of e to a power, the exponents are multiplied: $\left(e^k\right)^p = e^{pk}$

Proofs (Optional): See p.257 in the textbook.

Warning: We do **not** have nice rules for the log of a sum or a difference: $\ln\left(A \pm B\right)$

PART C: WHAT IF YOU FORGET THE RULES?

Experimenting with different powers of e may help you verify a guess to a rule you're not 100% confident about.

Example

Test the Product Rule: $\ln\left(AB\right) = \ln A + \ln B$

Let $A = e^2$ and $B = e^3$.

$$\ln\left(AB\right) = \ln A + \ln B$$
$$\ln\left(e^2 e^3\right) = \ln e^2 + \ln e^3$$
$$\ln\left(e^5\right) = \ln e^2 + \ln e^3$$
$$5 = 2 + 3$$
$$5 = 5 \qquad \left(\text{Checks out}\right)$$

This is not a proof, but it should be encouraging.
If your experiment does **not** work out, then you may have guessed the wrong rule!
Don't just make up your own rules!

PART D: MORE ON THE POWER RULE

The Power Rule for Logs is the most commonly abused rule among the three we've introduced thus far in this section.

In order for the rule, $\ln A^p = p \ln A$, to apply, we require the exponent, p, to apply to the **entire** base A, and **not** to the "log."

Examples

Yes or No: Are the two expressions equivalent, according to the Power Rule for Logs? (Assume $x > 0$ and $y > 0$.)

Expression #1	**Expression #2**	**Equivalent?**	**Comments**
$\ln xy^3$	$3 \ln xy$	**No**	The 3 does not apply to the x. Use the Product Rule, first.
$\ln (xy)^3$	$3 \ln xy$	**Yes**	The 3 applies to the entire xy base.
$(\ln x)^3$	$3 \ln x$	**No**	The 3 applies to the "ln," as well.
$\ln x^3$	$3 \ln x$	**Yes**	

Technical Note: In the rule $\ln A^p = p \ln A$, what if we allow $A < 0$? Then, it really matters what kind of number p is. For example, what if p is an even integer, say 2? Then, we have: $\ln A^2 = 2 \ln |A|$.

PART E: EXPANDING LOG EXPRESSIONS

Assume that all variables are restricted to positive values.

Example

Expand (i.e., completely expand, but evaluate expressions where appropriate):
$\ln \sqrt[3]{ex}$

Solution

$$\begin{aligned}
\ln \sqrt[3]{ex} &= \ln(ex)^{1/3} \\
&= \frac{1}{3}\ln(ex) && \text{(By the Power or "Smackdown" Rule)} \\
&= \frac{1}{3}(\ln e + \ln x) && \text{(By the Product Rule)} \\
&= \frac{1}{3}(1 + \ln x) && \text{(Evaluation)} \\
&\text{or} \quad \frac{1}{3} + \frac{1}{3}\ln x
\end{aligned}$$

Example

Expand: $\log_2 \frac{8x^3}{y}$

Solution

$$\log_2 \frac{8x^3}{y} = \log_2(8x^3) - \log_2 y$$

(You may skip the above step if you write the next step.)

$$\begin{aligned}
&= \log_2 8 + \log_2 x^3 - \log_2 y \\
&= \quad 3 \quad + 3\log_2 x - \log_2 y
\end{aligned}$$

(Evaluation and Power / "Smackdown" Rule)

Example

Expand: $\log \frac{a^2 b^3}{c^4 d}$

Solution

$$\log \frac{a^2 b^3}{c^4 d} = \left(\log a^2 + \log b^3\right) - \left(\log c^4 + \log d\right)$$

(You may skip the above step if you write the next step.)

$$= \log a^2 + \log b^3 - \log c^4 - \log d$$

(Warning: Watch out for that last minus sign! Remember that d was a factor of the **denominator** of the log argument.)

$$= 2\log a + 3\log b - 4\log c - \log d$$

In Calculus: These expansion techniques come in very handy when we do logarithmic differentiation (used to find the derivative of a complicated function) and when we differentiate complicated log functions. In the latter case, we apply the log rules to tear apart the log expression, and then we differentiate the pieces term-by-term. This is often easier than directly differentiating the given expression.

PART F: CONDENSING LOG EXPRESSIONS

Assume that all variables are restricted to positive values.

Example

Condense (i.e., completely condense): $\ln x + 3\ln y$

Solution

$$\ln x + 3\ln y = \ln x + \ln y^3 \quad (\text{"Reverse" Power / "Basketball" Rule})$$
$$= \ln\left(xy^3\right)$$

In the last step, we applied the Product Rule "in reverse":
The sum of the logs equals the log of the product.

Example

Condense (i.e., completely condense): $\frac{1}{2}\log 3x + \log y - 3\log z$

Solution

$$\frac{1}{2}\log 3x + \log y - 3\log z = \log\underbrace{(3x)^{1/2}}_{=\sqrt{3x}} + \log y - \log z^3$$

$(\text{"Reverse" Power / "Basketball" Rule})$

(Warning: Notice the grouping symbols around the $3x$.)

$$= \log\left(\sqrt{3x} \cdot y\right) - \log z^3$$

$(\text{"Reverse" Product Rule})$

$$= \log\frac{y\sqrt{3x}}{z^3}$$

(Warning: $\sqrt{3x}\,y$ may be confusing.)

In the last step, we applied the Quotient Rule "in reverse":
The difference of the logs equals the log of the quotient.

These tools will help us solve logarithmic equations in Section 3.4.

PART G: CHANGE OF BASE FORMULA

Example

We know that $\log_2 8 = 3$.

How can we approximate $\log_2 9$? We do not need a LOG_2 button on our calculators. It turns out the following formulas work:

$$\log_2 9 = \frac{\ln 9}{\ln 2} \approx 3.1699$$

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In fact, $\log_2 9 = \dfrac{\log_a 9}{\log_a 2}$ for **any** nice base *a*.

In general,

Change of Base Formula
If b is a nice base, and if $x > 0$, then
$\log_b x = \dfrac{\log_a x}{\log_a b}$, where a is **any** nice base.

Of course, we normally choose $a = e$ or $a = 10$ so that we can use the LN or the LOG button on our calculators.

In Calculus: The Change of Base Formula comes in very handy when differentiating log functions with various bases. We can then lean on the fact that the derivative of $\ln x$ is $\frac{1}{x}$.

Technical Note: In algorithm analysis, computer scientists often get very lazy when dealing with functions that model running time or space/memory requirements. Often, "a log is a log" to them.

Observe: the binary-friendly expression $\log_2 x = \frac{\ln x}{\ln 2} \quad \Rightarrow \quad \ln x = \underbrace{(\ln 2)}_{\text{constant}}(\log_2 x)$.

By this reasoning, the log functions for various bases are simply constant multiples of each other. These functions are lumped together in the category $O(\log x)$, where O (called "big-O") is referred to as "order."

Exponential functions such as $f(n) = 2^n$ grow much faster than log functions in the long run. If the running time (or space/memory requirements) of an algorithm is approximately exponential in n (the size of the input), then that is very bad news in the long run compared to an algorithm that is approximately logarithmic in n.

Proof of the Change of Base Formula (Optional):

Let $y = \log_b x$, where b is nice and $x > 0$.

$$\begin{aligned} \log_b x &= y \\ b^y &= x \\ \log_a b^y &= \log_a x \quad (\text{where } a \text{ is any nice base}) \\ y \log_a b &= \log_a x \\ y &= \frac{\log_a x}{\log_a b} \\ \log_b x &= \frac{\log_a x}{\log_a b} \end{aligned}$$