CHAPTER 4: TRIGONOMETRY (INTRO)

SECTION 4.1: (ANGLES); RADIAN AND DEGREE MEASURE

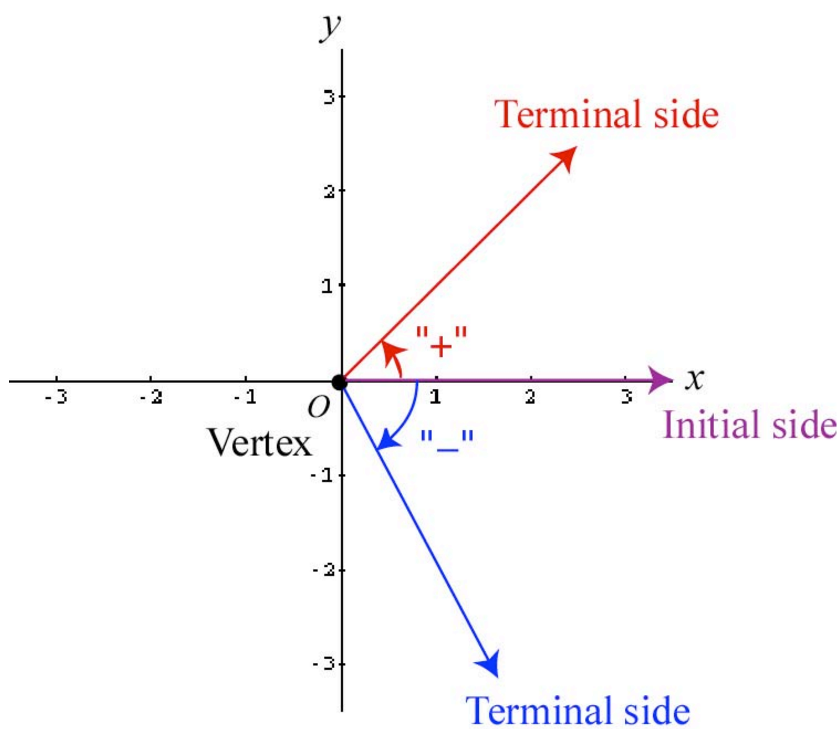
PART A: ANGLES

An angle is determined by rotating a ray (a "half-line") from an initial side to a terminal side about its endpoint, called the vertex.

A positive angle is determined by rotating the ray **counterclockwise**.

A negative angle is determined by rotating the ray **clockwise**.

A standard angle in standard position has the positive x-axis as its initial side and the origin as its vertex:



Angles are often denoted by capital letters (with maybe the $\angle$ symbol) and by Greek letters such as θ (theta), ϕ (phi), α (alpha), β (beta), and γ (gamma).

PART B: DEGREE MEASURE FOR ANGLES

We often associate angles with their rotational measures.

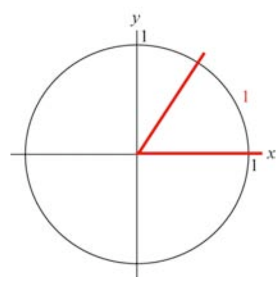
There are 360° (360 degrees) in a full (counterclockwise) revolution.
This is something of a cultural artifact; ancient Babylonians operated on a base-60 number system.

We sometimes use DMS (Degree-Minute-Second) measure instead of decimal degrees.
There are 60 minutes in 1 degree (Think: "hour"), and there are 60 seconds in 1 minute.
For example, $34^\circ 30' 20''$ denotes 34 degrees, 30 minutes, and 20 seconds.

PART C: RADIAN MEASURE FOR ANGLES

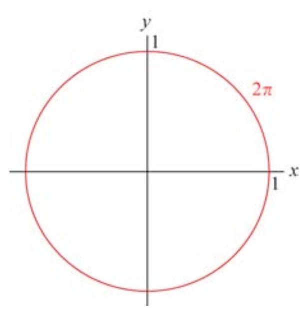
Radian measure is more "mathematically natural," and it is typically assumed in calculus.
In fact, radian measure is assumed if there are no units present.

Consider the unit circle (centered at the origin). 1 radian is defined to be the measure of a central angle (i.e., an angle whose vertex coincides with the center of the circle) that intercepts an arc length of 1 unit along the unit circle.



1 radian is about 57.3°; in fact, it is exactly $\left(\frac{180}{\pi}\right)^\circ$.

There are 2π radians in a full (counterclockwise) revolution, because the entire circle (which has circumference 2π) is intercepted exactly once by such an angle.



PART D: CONVERTING BETWEEN DEGREES AND RADIANS

2π radians is equivalent to 360°. Therefore, π radians is equivalent to 180°. Either relationship may be used to construct conversion factors.

In any unit conversion, we effectively multiply by 1 in such a way that the old unit is canceled out.

Example

Convert 45° into radians.

Solution

$$45^\circ = \left(45^\circ\right)\overbrace{\left(\frac{\pi\left[\text{rad}\right]}{180^\circ}\right)}^{=1} = \left(\cancel{45^\circ}\right)\left(\frac{\pi\left[\text{rad}\right]}{\cancel{180^\circ}_{\,4}}\right) = \frac{\boldsymbol{\pi}}{\mathbf{4}}\left[\textbf{rad}\right]$$

Example

Convert $\frac{5\pi}{18}$ radians into degrees.

Solution

$$\frac{5\pi}{18} = \left(\frac{5\pi}{18}\left[\text{rad}\right]\right)\underbrace{\left(\frac{180^\circ}{\pi\left[\text{rad}\right]}\right)}_{=1} = \left(\frac{5\cancel{\pi}}{\cancel{18}}\right)\left(\frac{\cancel{180^\circ}^{\,10^\circ}}{\cancel{\pi}}\right) = \mathbf{50^\circ}\text{, or}$$

$$\frac{5\pi}{18} = \frac{5}{18}\left(\pi\ \text{rad}\right) = \frac{5}{\cancel{18}}\left(\cancel{180^\circ}^{\,10^\circ}\right) = \mathbf{50^\circ}$$

Warning: Always make sure what mode your calculator is in (DEG vs. RAD) whenever you evaluate trig functions such as sin, cos, and tan.

PART E: ARC LENGTH

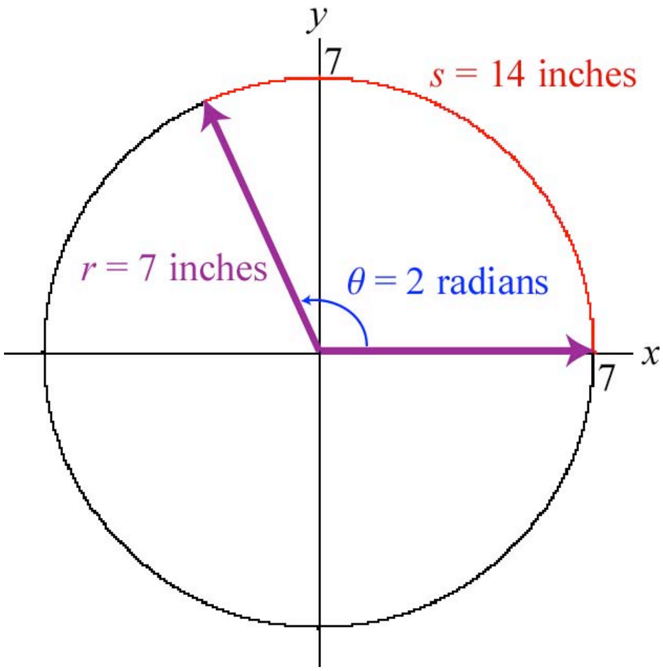
Consider a circle of radius r $(r > 0)$.
The arc length s of the arc intercepted by a central angle measuring θ radians is:

$$s = r\theta$$

Example

The arc length of an arc along a circle of radius 7 inches intercepted by a central angle measuring 2 radians is given by:

$$\begin{aligned} s &= r\theta \\ &= (7)(2) \\ &= \mathbf{14\ inches} \end{aligned}$$



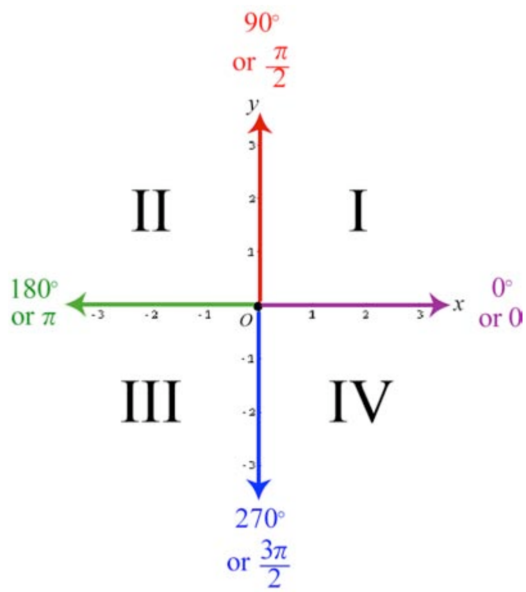
Note: If r is measured in inches, then s is measured in inches. Since θ is measured in radians, it may be seen as "unit-less." We can treat θ as simply a real value.

PART F: QUADRANTS AND QUADRANTAL ANGLES

The x- and y-axes divide the xy-plane into 4 quadrants.
Quadrant I is the upper right quadrant; the others are numbered in counterclockwise order.

A standard angle whose terminal side lies on the x- or y-axis is called a quadrantal angle.
Quadrantal angles correspond to "integer multiples" of 90° or $\frac{\pi}{2}$ radians.

The quadrants and some quadrantal angles:
(For convenience, we may label a standard angle by labeling its terminal side.)



Example

What quadrant does $\frac{5\pi}{6}$ lie in? (Through what quadrant does the terminal side pass when the angle is in standard position?)

Solution

Observe: $\underbrace{\frac{3\pi}{6}}_{=\frac{\pi}{2}} < \frac{5\pi}{6} < \underbrace{\frac{6\pi}{6}}_{=\pi}$, so $\frac{5\pi}{6}$ is in **Quadrant II**.

You could also consider degree measures: $90^\circ < 150^\circ < 180^\circ$

PART G: CLASSIFYING ANGLES

Type	Degree Measure	Radian Measure	Quadrant
Acute	in $(0^\circ, 90^\circ)$	in $\left(0, \frac{\pi}{2}\right)$	I
Right	90°	$\frac{\pi}{2}$	(Quadrantal)
Obtuse	in $(90^\circ, 180^\circ)$	in $\left(\frac{\pi}{2}, \pi\right)$	II

Complementary angles are a pair of positive angles that add up to 90°.
Supplementary angles are a pair of positive angles that add up to 180°.

Warning: It is easy to confuse these. Remember that "C" comes before "S" in the dictionary. Similarly, $90 < 180$.

PART H: COTERMINAL ANGLES

Standard angles that share the same terminal side are called coterminal angles. They differ by, at most, an integer number of full revolutions counterclockwise or clockwise.

If the angle θ is measured in **radians**, then its coterminal angles are of the form: $\theta + 2\pi n$, where n is any integer.

If the angle θ is measured in **degrees**, then its coterminal angles are of the form: $\theta + 360n^\circ$, where n is any integer.

Note: Since n could be negative, the "+" sign is sufficient in the above forms, as opposed to "$\pm$."

Example

The angles coterminal to $\frac{\pi}{3}$ are of the form $\frac{\pi}{3}+2\pi n$, where n is any integer.

To obtain some of these angles, we may take $\frac{\pi}{3}$ and successively add or subtract 2π, which equals $\frac{6\pi}{3}$ $\left(\text{Think: } 2=\frac{6}{3}\right)$:

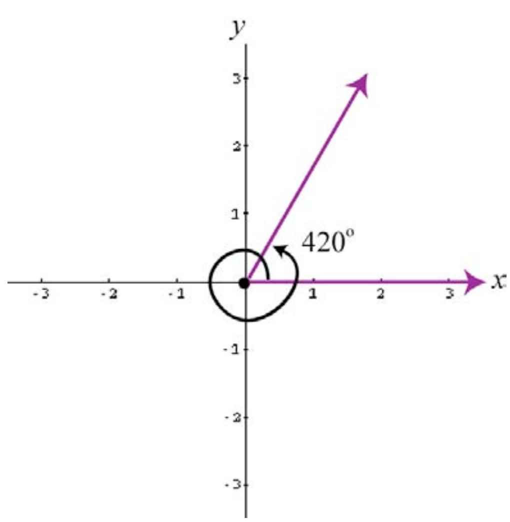
$$\ldots \quad -\frac{11\pi}{3} \xleftarrow{-\frac{6\pi}{3}} -\frac{5\pi}{3} \xleftarrow{-\frac{6\pi}{3}} \boxed{\frac{\pi}{3}} \xrightarrow{+\frac{6\pi}{3}} \frac{7\pi}{3} \xrightarrow{+\frac{6\pi}{3}} \frac{13\pi}{3} \quad \ldots$$

In degrees …

The angles coterminal to 60° are of the form $60^\circ+360n^\circ$, where n is any integer. To obtain some of these angles, we may take 60° and successively add or subtract 360°:

$$\ldots \quad -660^\circ \xleftarrow{-360^\circ} -300^\circ \xleftarrow{-360^\circ} \boxed{60^\circ} \xrightarrow{+360^\circ} 420^\circ \xrightarrow{+360^\circ} 780^\circ \quad \ldots$$

Below is a 420° angle:



Warning: If different scales are used for the x- and y-axes, there may be visual distortions. For example, a 45° angle may not "look like" a 45° angle, and perpendicular lines may not appear perpendicular. (Remember the "This is a square!" problem from Notes P.27?)