SECTIONS 4.2-4.4: TRIG FUNCTIONS  
(VALEUES AND IDENTITIES)

We will consider two general approaches: the Right Triangle approach, and the Unit Circle approach.

PART A: THE RIGHT TRIANGLE APPROACH

The Setup

The acute angles of a right triangle are complementary. Consider such an angle, $\theta$. Relative to $\theta$, we may label the sides as follows:

![Diagram of a right triangle with labels]

The hypotenuse always faces the right angle, and it is always the longest side.

The other two sides are the legs. The opposite side (relative to $\theta$) faces the $\theta$ angle. The other leg is the adjacent side (relative to $\theta$).

We sometimes use the terms “hypotenuse,” “leg,” and “side” when we are actually referring to a length.
Defining the Six Basic Trig Functions (where $\theta$ is acute)

![Diagram of a right triangle with labels for hypotenuse, opposite side, and adjacent side.

The Ancient Curse (or “How to Define Trig Functions”)

SOH-CAH-TOA

Sine $\theta = \sin \theta = \frac{\text{Opp.}}{\text{Hyp.}}$

Cosine $\theta = \cos \theta = \frac{\text{Adj.}}{\text{Hyp.}}$

Tangent $\theta = \tan \theta = \frac{\text{Opp.}}{\text{Adj.}}$

Reciprocal Identities (or “How to Define More Trig Functions”)

Cosecant $\theta = \csc \theta = \frac{1}{\sin \theta} = \frac{\text{Hyp.}}{\text{Opp.}}$

Secant $\theta = \sec \theta = \frac{1}{\cos \theta} = \frac{\text{Hyp.}}{\text{Adj.}}$

Cotangent $\theta = \cot \theta = \frac{1}{\tan \theta} = \frac{\text{Adj.}}{\text{Opp.}}$

**Warning:** Remember that the reciprocal of $\sin \theta$ is $\csc \theta$, not $\sec \theta$.

**Note:** We typically treat “0” and “undefined” as reciprocals when we are dealing with trig functions. Your algebra teacher will not want to hear this, though!
Quotient Identities

We may also define $\tan \theta$ and $\cot \theta$ as follows:

\[
\begin{align*}
\tan \theta &= \frac{\sin \theta}{\cos \theta} \\
\cot \theta &= \frac{\cos \theta}{\sin \theta}
\end{align*}
\]

Why is this consistent with SOH-CAH-TOA?

\[
\begin{align*}
\sin \theta &= \frac{\text{Opp.}}{\text{Hyp.}} = \frac{\text{Opp.}}{\text{Adj.}} = \frac{\text{Opp.}}{\text{Adj.}} = \tan \theta \\
\cos \theta &= \frac{\text{Adj.}}{\text{Hyp.}} \\
\cot \theta &= \frac{1}{\tan \theta}
\end{align*}
\]

cot $\theta$ is the reciprocal of $\tan \theta$.

We will discuss the Cofunction Identities soon ….
The Pythagorean Theorem

Given two sides, you can find the length of the third by using the Pythagorean Theorem (see p.349 for one of many proofs): 

\[ (\text{Opp.})^2 + (\text{Adj.})^2 = (\text{Hyp.})^2 \]

Pythagorean Triples

Pythagorean triples are a set of three integers that can represent the side lengths of a right triangle.

The most famous Pythagorean triples are:

- 3-4-5
- 5-12-13
- 8-15-17

Some less famous ones are:

- 7-24-25
- 9-40-41

Warning: Remember that the hypotenuse must be the longest side. If the two legs of a right triangle have lengths 3 and 5, the hypotenuse is not 4.

Similar triangles have the same shape (but perhaps not the same size), and they share the same three angles. Their corresponding side lengths are in the same proportion. By considering similar triangles, we can get some more Pythagorean triples (though these are not “primitive,” as the ones above were). For example, 6-8-10 triangles, 9-12-15 triangles, and so forth, are similar to 3-4-5 triangles.

Technical Note: There are infinitely many primitive Pythagorean triples. This has been determined in the field of number theory.

The more Pythagorean triples you know, the less frequently you will have to crank through the Pythagorean Theorem.
Example

Let $\theta$ be an acute angle such that $\cos\theta = \frac{4}{7}$. Find all six basic trig values for $\theta$.

Solution

We will sketch a right triangle model. Either of the two models below would work, but the one on the left will probably be easier to work with.

A Pythagorean triple is not evident here, so we will use the Pythagorean Theorem to find $x$.

\[ x^2 + (4)^2 = (7)^2 \]
\[ x^2 + 16 = 49 \]
\[ x^2 = 33 \]
\[ x = \pm\sqrt{33} \]

Take $x = \sqrt{33}$

If $\theta$ is acute, then we always label our sides with positive numbers. Later, we will sometimes use negative numbers.

Find the six basic trig values for $\theta$:

\[ \sin \theta = \frac{\text{Opp.}}{\text{Hyp.}} = \frac{\sqrt{33}}{7} \]
\[ \cos \theta = \frac{\text{Adj.}}{\text{Hyp.}} = \frac{4}{7} \quad \text{(Given)} \]
\[ \tan \theta = \frac{\text{Opp.}}{\text{Adj.}} = \frac{\sqrt{33}}{4} \]

\[ \Rightarrow \text{Reciprocals} \]

\[ \csc \theta = \frac{7}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = \frac{7\sqrt{33}}{33} \]
\[ \sec \theta = \frac{7}{4} \]
\[ \cot \theta = \frac{4}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}} = \frac{4\sqrt{33}}{33} \]
Special Triangles

We can use these triangles to find the basic trig values for 30°, 45°, and 60°. Later, we will efficiently summarize these values in a table.

45°-45°-90° Triangles

These are also known as isosceles right triangles, because they have two congruent (same-length) sides. They represent a family of similar triangles.

The most famous such triangle is:

(The tick marks on the legs indicate that they are congruent.)

The \( \sqrt{2} \) is obtained from the Pythagorean Theorem.

**Historical Note:** The fact that the hypotenuse of this triangle is irrational really freaked out the cult of ancient Pythagoreans. One member who leaked the secret was done in by the cult.

Observe that both \( \sin 45° \) and \( \cos 45° \) are \( \frac{1}{\sqrt{2}} \), or \( \frac{\sqrt{2}}{2} \).

Then, both \( \csc 45° \) and \( \sec 45° \) are \( \sqrt{2} \).

**Note:** It is sometimes easier if we do not rationalize a denominator before we take a reciprocal.

Observe that both \( \tan 45° \) and \( \cot 45° \) are 1.

In general, the hypotenuse of any 45°-45°-90° triangle is \( \sqrt{2} \) times either leg.
This is another family of similar triangles.

An equilateral triangle is a triangle in which all three sides are congruent and all three interior angles are $60^\circ$.

We will start with an equilateral triangle of side length 2. We will draw a perpendicular from one vertex to the opposing side, thus bisecting (cutting in half) both the angle at the vertex and the opposing side.

The $\sqrt{3}$ is obtained from the Pythagorean Theorem.

Remember that, in a triangle, longer sides face larger angles. Here, the shortest side must face the $30^\circ$ angle.

In general, the hypotenuse is twice as long as the shortest side of any $30^\circ\text{-}60^\circ\text{-}90^\circ$ triangle, and the medium side is $\sqrt{3}$ times as long as the shortest side.

Observe that both $\sin 30^\circ$ and $\cos 60^\circ$ are $\frac{1}{2}$. This is explained by the …
Cofunction Identities

Cofunctions are short for “complementary [trig] functions.” sin and cos are a pair of cofunctions, as are tan and cot, and also sec and csc. Read out their full names!

Cofunctions of complementary angles are equal.

Remember that the acute angles of a right triangle are complementary.

For example, why is $\sin 30^\circ = \cos 60^\circ$? When we shift our perspective from one acute angle of a right triangle to the other, the “adjacent side” and the “opposite side” are switched.

In fact, we can look beyond right triangles ….

Cofunction Identities

If $\theta$ is measured in radians, then for all real values of $\theta$:

$\sin \theta = \cos \left( \frac{\pi}{2} - \theta \right)$

$\cos \theta = \sin \left( \frac{\pi}{2} - \theta \right)$

If $\theta$ is measured in degrees, then for all angles $\theta$:

$\sin \theta = \cos \left( 90^\circ - \theta \right)$

$\cos \theta = \sin \left( 90^\circ - \theta \right)$

We have analogous relationships for tan and cot, and for sec and csc.
PART B: THE UNIT CIRCLE APPROACH

The Setup

Consider a standard angle $\theta$ measured in radians (or, equivalently, let $\theta$ represent a real number).

The point $P(\cos \theta, \sin \theta)$ is the intersection point between the terminal side of the angle and the unit circle. The slope of the terminal side is, in fact, $\tan \theta$.

Note: The intercepted arc along the circle (in red) has arc length $\theta$.

The figure below demonstrates how this is consistent with the SOH-CAH-TOA (or Right Triangle) approach. Observe: $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\text{rise}}{\text{run}} = \text{slope of terminal side}$

For all real $\theta$,

$-1 \leq \cos \theta \leq 1$
$-1 \leq \sin \theta \leq 1$
$\tan \theta$ can be any real number, or it can be undefined
“THE Table”

We will use our knowledge of the $30^\circ$-$60^\circ$-$90^\circ$ and $45^\circ$-$45^\circ$-$90^\circ$ special triangles to construct THE Table below. The unit circle approach is used to find the trig values for quadrantal angles such as $0^\circ$ and $90^\circ$.

Let $P$ be the intersection point between the terminal side of the standard angle $\theta$ and the unit circle. $P$ has coordinates $(\cos \theta, \sin \theta)$.

<table>
<thead>
<tr>
<th>Key Angles $\theta$: Degrees, (Radian)</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\tan \theta = \frac{\sin \theta}{\cos \theta}$</th>
<th>Intersection Point $P(\cos \theta, \sin \theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$, (0)</td>
<td>$\frac{\sqrt{0}}{2} = 0$</td>
<td>1</td>
<td>$\frac{0}{1} = 0$</td>
<td>$(1, 0)$</td>
</tr>
<tr>
<td>$30^\circ$, $\left(\frac{\pi}{6}\right)$</td>
<td>$\frac{\sqrt{1}}{2} = \frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$</td>
<td>$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$</td>
</tr>
<tr>
<td>$45^\circ$, $\left(\frac{\pi}{4}\right)$</td>
<td>$\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$</td>
<td>$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$</td>
</tr>
<tr>
<td>$60^\circ$, $\left(\frac{\pi}{3}\right)$</td>
<td>$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}/2}{1/2} = \sqrt{3}$</td>
<td>$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$</td>
</tr>
<tr>
<td>$90^\circ$, $\left(\frac{\pi}{2}\right)$</td>
<td>$\frac{\sqrt{4}}{2} = 1$</td>
<td>0</td>
<td>$\frac{1}{0}$ is undefined</td>
<td>$(0, 1)$</td>
</tr>
</tbody>
</table>

Warning: $\frac{\pi}{5}$ is not a “special” angle.

The values for the reciprocal functions, $\csc \theta$, $\sec \theta$, and $\cot \theta$, are then readily found. Remember that it is sometimes better to take a trig value where the denominator is not rationalized before taking its reciprocal.

For example, because $\tan 30^\circ = \frac{1}{\sqrt{3}}$, we know immediately that $\cot 30^\circ = \sqrt{3}$.
Observe:

- The pattern in the “sin” column

  **Technical Note:** An explanation for this pattern appears in the Sept. 2004 issue of the College Mathematics Journal (p.302).

- The fact that the “sin” column is flipped (or reversed) to form the “cos” column. This is due to the Cofunction Identities (or the Pythagorean Identities, which we will cover).

- As $\theta$ increases from $0^\circ$ to $90^\circ$ (i.e., from 0 to $\frac{\pi}{2}$ radians),
  
  - $\sin \theta$ (the $y$-coordinate of $P$) increases from 0 to 1.

    **Note:** This is more obvious using the Unit Circle approach instead of the Right Triangle approach.

  - $\cos \theta$ (the $x$-coordinate of $P$) decreases from 1 to 0.

  - $\tan \theta$ (the slope of the terminal side of the standard angle $\theta$) starts at 0, increases, and approaches $\infty$.

- The table is consistent with our observations based on the special triangles.

  For example, consider $\theta = 60^\circ$ \left( or $\frac{\pi}{3}$ \right):

  ![Diagram of a right triangle and unit circle showing a point P(1/2, √3/2) at 60° and 30°, with y-axis labeled 1, 2, and x-axis labeled 1/2, 2.](image)
• Here is the “Big Picture.” Remember that each intersection point is of the form \( P(\cos \theta, \sin \theta) \).

- The blue point for the 45\(^\circ\) angle has coordinates \( \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \); it lies on the line \( y = x \), which has slope (i.e., tangent) 1.

- Quick – what is \( \cos 60^\circ \)? It is \( \frac{1}{2} \), as opposed to \( \frac{\sqrt{3}}{2} \).

The red point for the 60\(^\circ\) angle has coordinates \( \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \); it is “high” and “to the left” relative to the blue and green points. Its \( x \)-coordinate (which corresponds to \( \cos \)) must be relatively low, then. Granted, \( \frac{1}{2} \), which is equal to 0.5, is not really \textbf{that} low, but it is lower than \( \frac{\sqrt{3}}{2} \).
Other Tricks:

• You may remember that \( \tan 45^\circ = 1 \), and that \( \frac{\sqrt{3}}{3} \) and \( \sqrt{3} \) are other key tan values. Which is \( \tan 30^\circ \), and which is \( \tan 60^\circ \)? Remember that \( \tan \theta \) corresponds to the slope of the terminal side, so the higher slope \( \left( \frac{\sqrt{3}}{3} \right) \) must be \( \tan 60^\circ \).

• Here is a nice way of remembering key trig values for \( 90^\circ \) (or \( \frac{\pi}{2} \) radians); thanks to some clever students for this:

\[
\begin{array}{ccc}
\text{SIN} & \text{COS} & \text{TAN} \\
\hline \\
\end{array}
\]

\[
\sin \frac{\pi}{2} = 1 , \text{ whose Roman numeral is I.} \\
\cos \frac{\pi}{2} = 0 , \text{ which resembles the letter “O.”} \\
\tan \frac{\pi}{2} \text{ is undefined, so there will be a vertical “A”symptote at} \\
x = \frac{\pi}{2} \text{ when we graph } f(\theta) = \tan \theta .
\]