Example

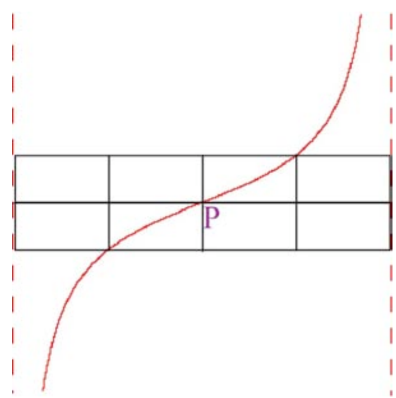
Use the Frame Method to graph one cycle of the graph of
$y = 2\tan\left(\frac{2}{5}x\right) - 3$. (There are infinitely many possible cycles.)

Solution

Fortunately, $b = \frac{2}{5} > 0$. If $b < 0$, we would need to use the Even/Odd Properties. Remember that both tan and cot are odd functions.

Pivot: $(p = 0, d = -3)$

Cycle shape: We have a tan graph with $a = 2 > 0$, so we will use:

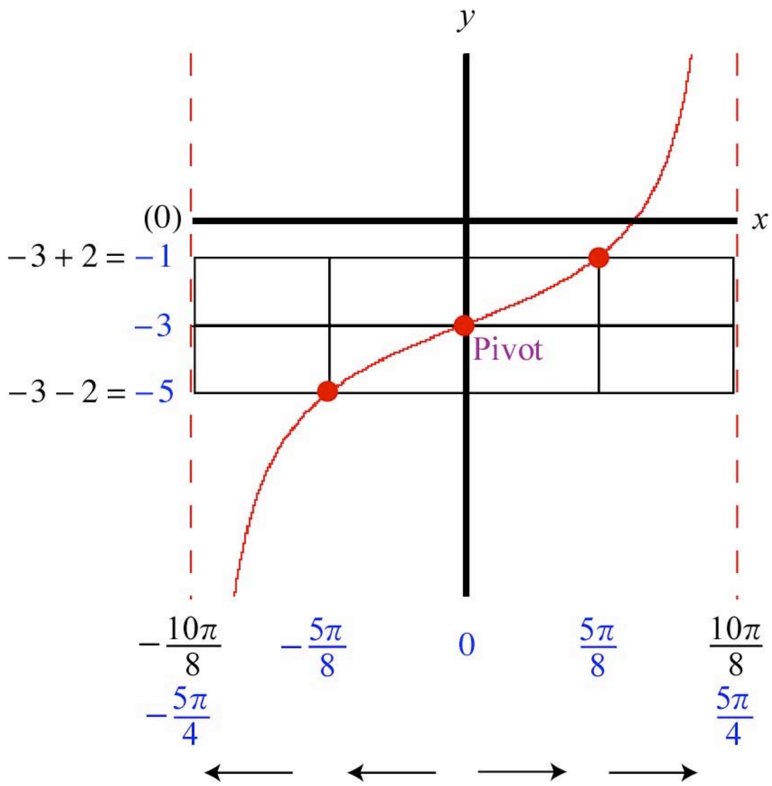


"**A**mplitude" $= |a| = |2| = 2$

Period $= \frac{\pi}{b} = \frac{\pi}{2/5} = \frac{5\pi}{2}$

Increment $= \frac{1}{4}(\text{Period}) = \frac{1}{4}\left(\frac{5\pi}{2}\right) = \frac{5\pi}{8}$

The Frame



Since there was no discernible phase shift, we see some nice symmetry between the positive and negative *x*-coordinates on this tan frame.

Note: If you would prefer to start the labeling process at the "left-center" point, just as for sin and cos cycles, you could find the *x*-coordinate of a VA by setting the argument of tan equal to $-\frac{\pi}{2}$ or $\frac{\pi}{2}$ (for example), which correspond to asymptotes for $y = \tan x$, and solving for *x*. Here:

$$\frac{2}{5}x = -\frac{\pi}{2}$$

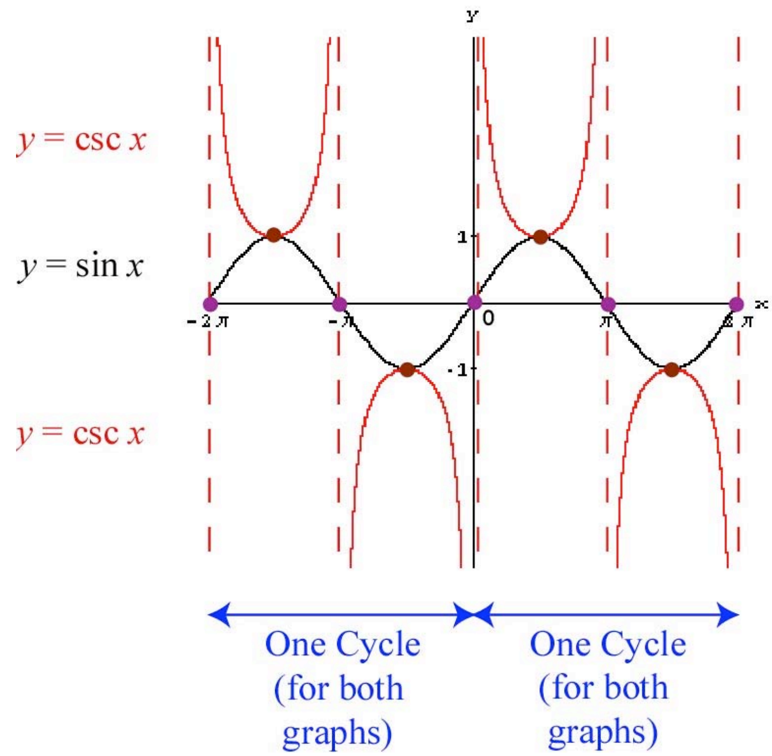
$$x = \left(\frac{5}{2}\right)\left(-\frac{\pi}{2}\right)$$

$$x = -\frac{5\pi}{4}$$

If you are dealing with a cot graph, then you would set the argument of cot equal to 0, just as for sin and cos graphs.

PART G: GRAPHS OF CSC AND SEC FUNCTIONS ("UP-U, DOWN-U" GRAPHS)

Remember that $\csc x = \dfrac{1}{\sin x}$.



How can we use the graph of $y = \sin x$ to obtain the graph of $y = \csc x$?

1) Draw VAs through the x-intercepts (in purple) of the $\sin x$ graph.

2) Between any consecutive pair of VAs:

If the $\sin x$ graph lies **above** the x-axis, then draw an **"up-U"** that has as its **minimum** point (in brown) the **maximum** point of the sin graph and that approaches both VAs.

If the $\sin x$ graph lies **below** the x-axis, then draw a **"down-U"** that has as its **maximum** point (also in brown) the **minimum** point of the sin graph and that approaches both VAs.

Warning: When graphing $\csc x$ and $\sec x$, the VAs separate the graphs into **half cycles**, not full cycles.

Why 1)?

Observe that $\csc x$ is undefined $\Leftrightarrow$ $\sin x = 0$.

Why 2)?

The reciprocal of 1 is 1, so: $\csc x = 1 \Leftrightarrow \sin x = 1$. This explains the brown intersection points between the "up-U"s and the $\sin x$ graph.

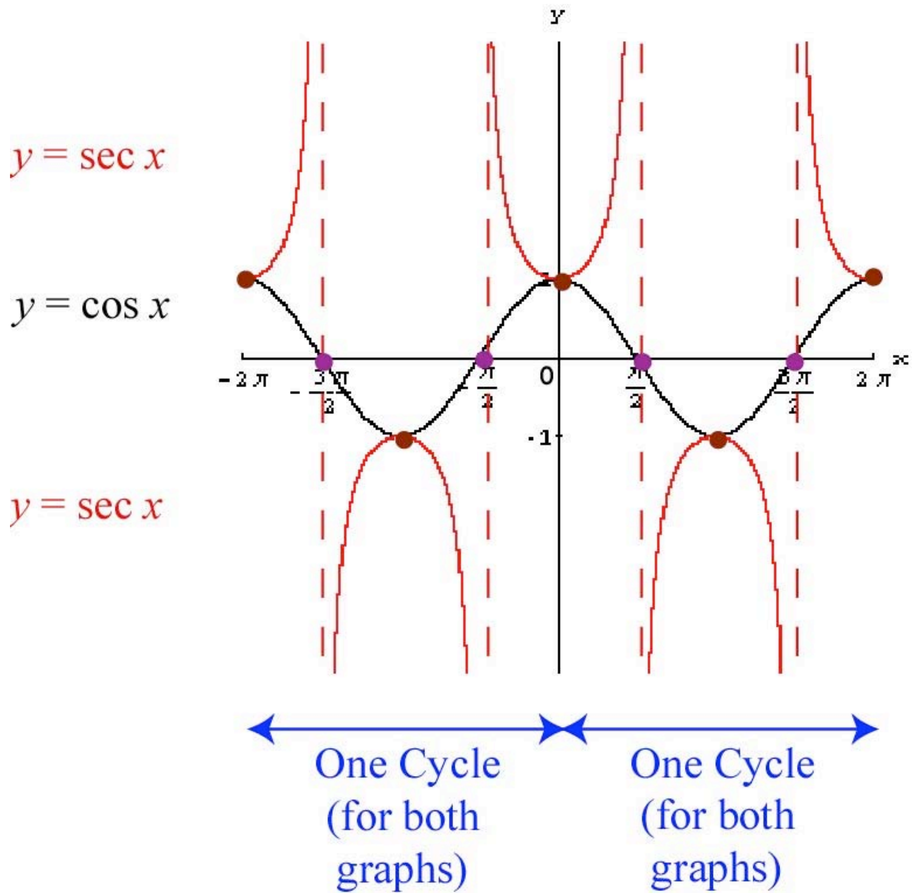
Likewise, the reciprocal of -1 is -1, so: $\csc x = -1 \Leftrightarrow \sin x = -1$. This explains the brown intersection points between the "down-U"s and the $\sin x$ graph.

Because $\sin x$ and $\csc x$ are reciprocal trig functions, we know that, between the asymptotes of the $\csc x$ graph, one increases $\Leftrightarrow$ the other decreases. For example, as $\sin x$ decreases from 1 to $\frac{1}{2}$, $\csc x$ actually increases from 1 to 2. As $\sin x$ decreases from 1 to 0, $\csc x$ increases without bound; it approaches ∞.

Technical Note: Another reason why we get "U"-shaped structures is that each "sine lump" between a pair of consecutive x-intercepts is symmetric about it maximum/minimum point. This leads to the symmetry of the "U" shapes.

The same procedure is applied to the graph of $y = \cos x$ to obtain the graph of $y = \sec x$.

Remember that $\sec x = \dfrac{1}{\cos x}$.



Because reciprocal periodic functions have the same period, the $\csc x$ and $\sec x$ functions have period 2π (because that is the period of the $\sin x$ and $\cos x$ functions).

One cycle of the $\csc x$ graph or the $\sec x$ graph must include exactly one "up-U," exactly one "down-U," and nearby VAs. One of the "U"s may be broken up into pieces at the left and right edges of the cycles; in other words, "wraparounds" are permissible when "counting" the "U"s.

Observe that:

- Reciprocal functions have the same sign (where both are defined), and this property is satisfied by the "up-U, down-U" structure of our $\csc x$ and $\sec x$ graphs.

- Because the $\sin x$ function is odd, the $\csc x$ function is odd, also. The $\csc x$ graph is symmetric about the origin.

- Because the $\cos x$ function is even, the $\sec x$ function is even, also. The $\sec x$ graph is symmetric about the *y*-axis.

- The graphs of $y = \cot x \left(= \dfrac{\cos x}{\sin x} \right)$ and $y = \csc x \left(= \dfrac{1}{\sin x} \right)$ have the same set of VAs (corresponding to where $\sin x$ is 0) and, therefore, the same domain for their corresponding functions. (See next page.)

- Similarly, the graphs of $y = \tan x \left(= \dfrac{\sin x}{\cos x} \right)$ and $y = \sec x \left(= \dfrac{1}{\cos x} \right)$ have the same VAs (corresponding to where $\cos x$ is 0) and the same domain for their corresponding functions. (See next page.)

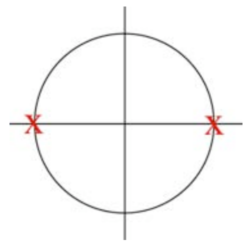
- $\csc x$ and $\sec x$ are never 0 in value. Their graphs have no *x*-intercepts.

PART H: DOMAIN AND RANGE

Just as for $\cot x$, the domain for the $\csc x$ function is:

$$\left\{ x \mid x \neq \pi n \text{ for all integers } n \right\}.$$

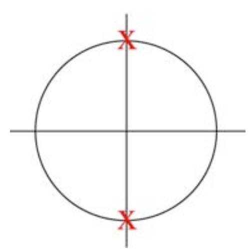
Remember: The zeros of $\sin x$ are excluded from the domain.



Just as for $\tan x$, the domain for the $\sec x$ function is:

$$\left\{ x \;\middle|\; x \neq \frac{\pi}{2} + \pi n \;\left(\text{i.e., } x \neq \underbrace{(2n+1)}_{\text{an odd integer}} \frac{\pi}{2} \right) \text{ for all integers } n \right\}.$$

Remember: The zeros of $\cos x$ are excluded from the domain.



The range for both the $\csc x$ and the $\sec x$ functions is: $(-\infty, -1] \cup [1, \infty)$

Think: A "high-low" game where you "win" if you get a value that's low (i.e., -1 or lower) or high (i.e., 1 or higher).

Think: We're turning the interval $[-1, 1]$ (the range for the $\sin x$ and $\cos x$ functions) "inside out."

The above range does not contain 0.
The fact that $\csc x$ and $\sec x$ are never 0 in value proves helpful!

PART I: TRANSFORMATIONS

To graph $y = a\csc\left[b(x-p)\right] + d$, first graph $y = a\sin\left[b(x-p)\right] + d$, and apply the "up-U, down-U" trick to it.

To graph $y = a\sec\left[b(x-p)\right] + d$, first graph $y = a\cos\left[b(x-p)\right] + d$, and apply the "up-U, down-U" trick to it.

If there is a vertical shift (i.e., if $d \neq 0$), then the VAs will pass through the "midpoints" (or inflection points) of the corresponding sin or cos graph; those points are no longer x-intercepts, however.

Technical Note: The "up-U, down-U" trick works to get us from $y = a\sin(bx)$, say, to $y = a\csc(bx)$, because the observations we made earlier still hold.
If both graphs are translated horizontally by p units and vertically by d units, we see that the same trick works to get us from $y = a\sin\left[b(x-p)\right] + d$ to $y = a\csc\left[b(x-p)\right] + d$. We can envision the "up-U"s and "down-U"s (and nearby VAs) following the sin graph to its new position. Basically, the transformations that affect the sin graph simultaneously affect the csc graph. Make sure that you consider the translations **after** the nonrigid vertical and horizontal transformations, though!

PART J: EXAMPLE

Use the Frame Method to graph one cycle of the graph of
$y = -3\sec\left(4x + \pi\right) + 2$.

Solution

Fortunately (OK OK, I set it up), we did $y = -3\cos\left(4x + \pi\right) + 2$ in the Section 4.5 Notes: 4.50-4.52. The cycle for that graph is in red.

To obtain a cycle for our desired graph (in purple), we:

1) Draw VAs through the midpoints (i.e., inflection points) of the cos graph.

   Warning: In this Example, there is a vertical shift, so do **not** draw VAs through the x-intercepts of the cos graph.

2) Draw the "up-U" and "down-U" as described in Notes 4.65. This gives one complete cycle of our sec graph. However, because we are dealing with a sec graph, one of the "U"s will be broken into halves at the edges of the frame. (This is not true for a csc graph, if you use our usual method.)

You may want to lengthen the y-axis to make the picture look nicer.