SECTION 4.7: INVERSE TRIG FUNCTIONS

You may want to review Section 1.8 on inverse functions.

PART A: GRAPH OF $\sin^{-1} x$ (or $\arcsin x$)

Warning: Remember that f^{-1} denotes function inverse, not multiplicative inverse (or reciprocal). Usually, $f^{-1} \neq \frac{1}{f}$. In particular, $\sin^{-1} x \neq \frac{1}{\sin x}$, or $\csc x$. We **can** say that $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$. Although it is often helpful in Calculus to rewrite $\sin^n x$ as $(\sin x)^n$, this is **not** true of $\sin^{-1} x$, because -1 is **not** an exponent in that case. However, -1 **does** act as an exponent in $(\sin x)^{-1}$.

If $f(x) = \sin x$, and the domain is **R** (which is, after all, the implied domain), then f is **not** a one-to-one function, and it has no inverse **function**.

We want to define an inverse sine (or "arcsine") function $f^{-1}(x) = \sin^{-1} x$ (or $\arcsin x$). To do so, we must restrict the domain of $f(x) = \sin x$ so that it is a one-to-one function whose graph passes the HLT (Horizontal Line Test).

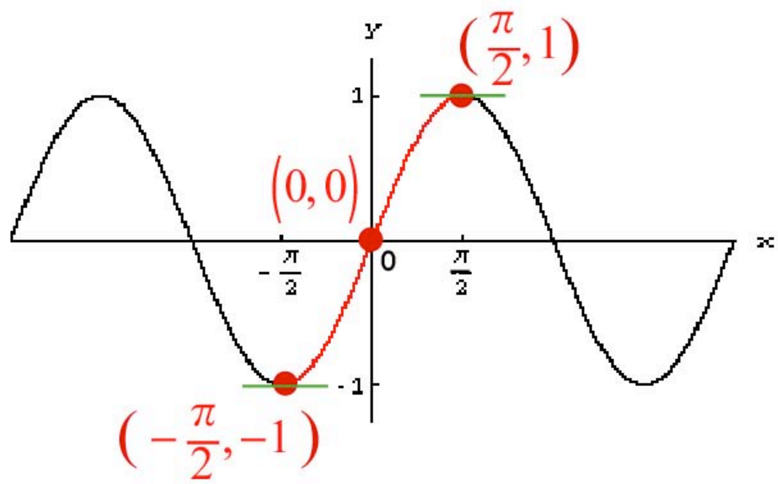
What should this restricted domain be? It should be an x-interval on which the $\sin x$ graph:

1) Passes the HLT, and

2) Is as "tall" as the original, unrestricted $\sin x$ graph. In other words, we would like the range to be the same as before.

It is universally agreed that we take the x-interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as our restricted domain. The resulting range for our $\sin x$ function remains $[-1, 1]$.

The resulting graph is in red below:
(The *x*- and *y*-axes are scaled differently.)



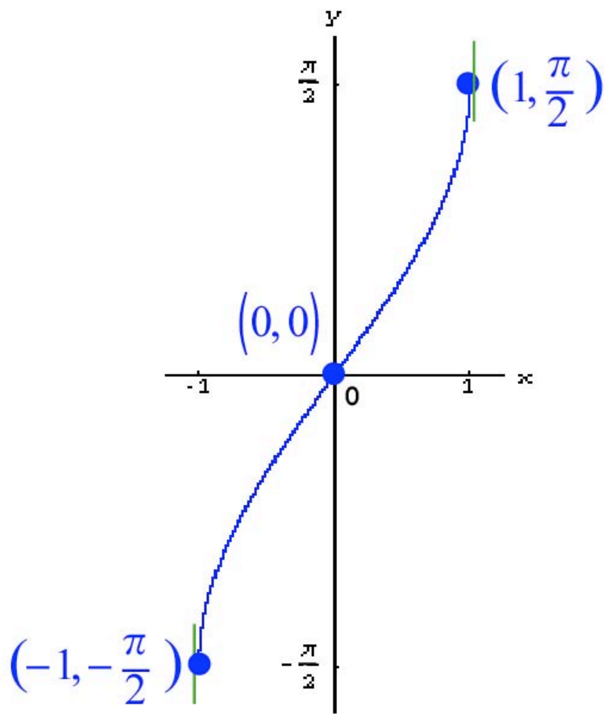
Observe that:

- The function increases on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

- The graph switches from concave up to concave down at $(0,0)$.

  It may be easier to remember that the graph is a snake of finite length that has horizontal (one-sided) tangent lines (in green) at its endpoints.

The graph of $f^{-1}(x) = \sin^{-1} x$ (or $\arcsin x$), the arcsine function, is obtained by switching the *x*- and *y*-coordinates of all the points on the red graph we just saw. (Reflecting the red graph about the line $y = x$ may be hard to visualize.) We obtain:



Observe that:

- The inverse function also increases, but on the interval $[-1, 1]$.
  The three indicated points above suggest this.

- However, the graph switches from concave down to concave up at $(0, 0)$.
  It may be easier to remember that the graph is a snake of finite length that has vertical (one-sided) tangent lines (in green) at its endpoints.

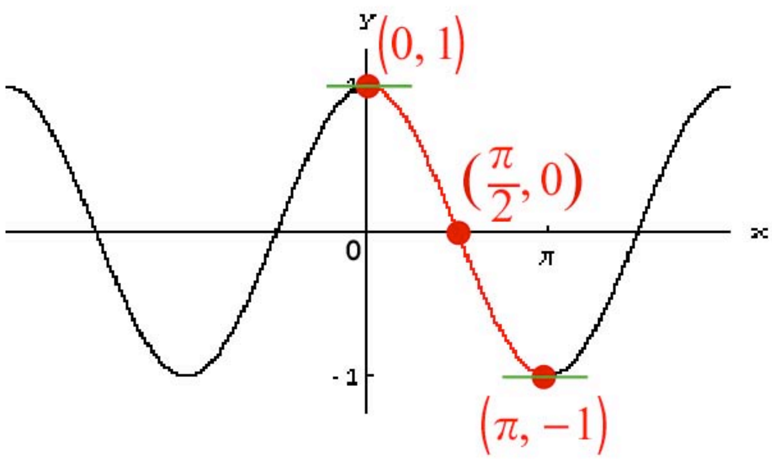
Remember that, for a pair of inverse functions, the domain of one is the range of the other.

	Domain	**Range**
$\sin x$ (restricted)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$\sin^{-1} x$ (or $\arcsin x$)	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

PART B : GRAPH OF $\cos^{-1} x$ (or $\arccos x$)

It is universally agreed that we take the x-interval $\left[0, \pi\right]$ as our restricted domain for $f(x) = \cos x$. The resulting range remains $\left[-1, 1\right]$.

The resulting graph is in red below:
(The x- and y-axes are scaled differently.)

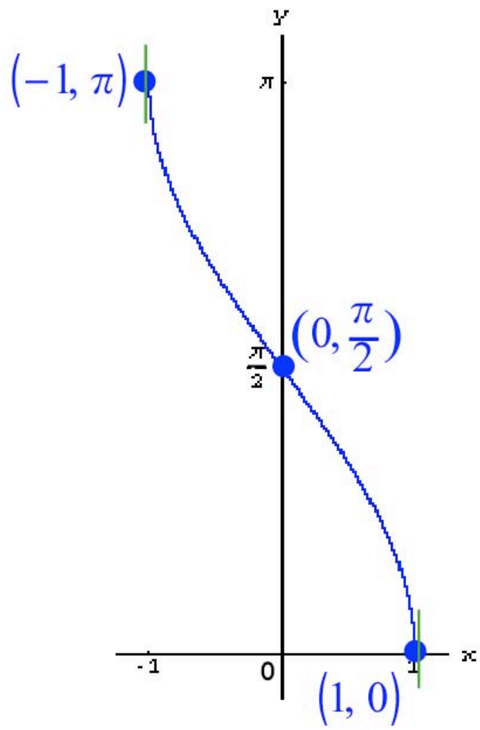


Observe that:

- The function decreases on the interval $\left[0, \pi\right]$.

- The graph switches from concave down to concave up at the "midpoint" $\left(\frac{\pi}{2}, 0\right)$. It may be easier to remember that, just as for $\sin x$, the graph is a snake of finite length that has horizontal (one-sided) tangent lines (in green) at its endpoints.

The graph of $f^{-1}(x) = \cos^{-1} x$ (or $\arccos x$), the arccosine function, is obtained by switching the x- and y-coordinates of all the points on the red graph we just saw. We obtain:



Observe that:

- The inverse function also decreases, but on the interval $[-1, 1]$.
  The three indicated points above suggest this.

- However, the graph switches from concave up to concave down at the "midpoint" $\left(0, \frac{\pi}{2}\right)$. It may be easier to remember that, just as for $\sin^{-1} x$, the graph is a snake of finite length that has vertical (one-sided) tangent lines (in green) at its endpoints.

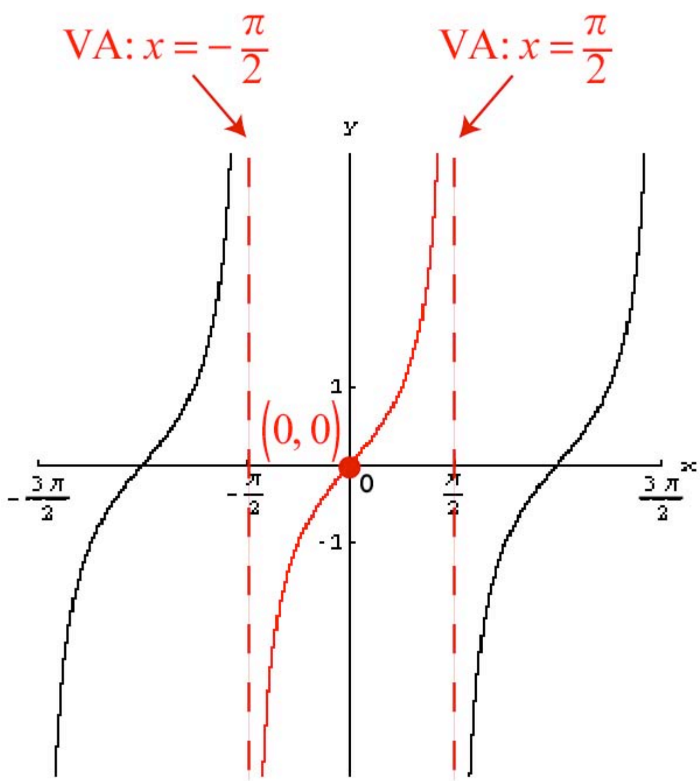
Remember that, for a pair of inverse functions, the domain of one is the range of the other.

	Domain	**Range**
$\cos x$ (restricted)	$[0, \pi]$	$[-1, 1]$
$\cos^{-1} x$ (or $\arccos x$)	$[-1, 1]$	$[0, \pi]$

PART C : GRAPH OF $\tan^{-1} x$ (or $\arctan x$)

It is universally agreed that we take the **open** x-interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as our restricted domain for $f(x) = \tan x$. The resulting range remains **R**, or $(-\infty, \infty)$.

The resulting graph is in red below:
(The x- and y-axes are scaled differently.)

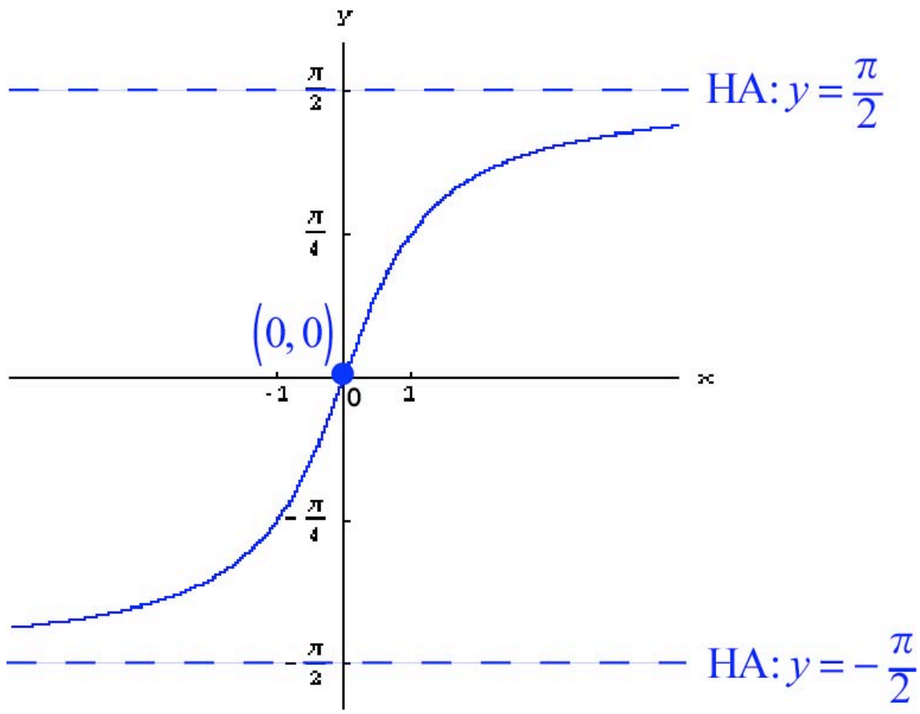


Observe that:

- The function increases on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- The graph switches from concave down to concave up at $(0, 0)$.

  It may be easier to remember that the graph is a snake of **infinite** length (unlike before) that approaches the **vertical** asymptotes (VAs) at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

The graph of $f^{-1}(x) = \tan^{-1} x$ (or $\arctan x$), the arctangent function, is obtained by switching the x- and y-coordinates of all the points on the red graph we just saw. We obtain:



Observe that:

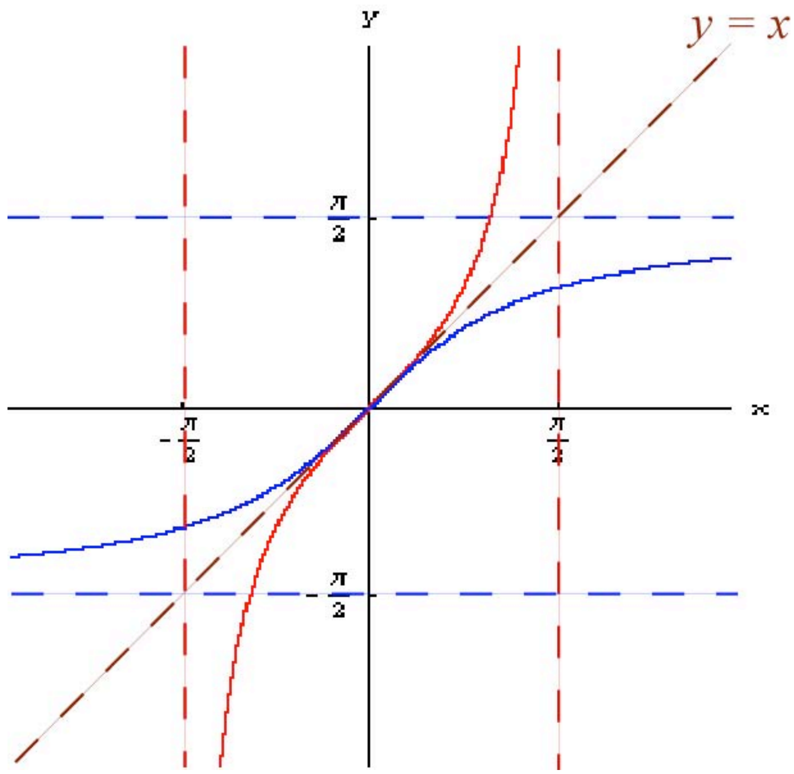
- The inverse function also increases, but on all of **R**.
- However, the graph switches from concave up to concave down at $(0, 0)$.

  It may be easier to remember that the graph is a snake of **infinite** length that approaches the **horizontal** asymptotes (HAs) at $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.

Remember that, for a pair of inverse functions, the domain of one is the range of the other.

	Domain	**Range**
$\tan x$ (restricted)	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	**R**, or $(-\infty, \infty)$
$\tan^{-1} x$ (or $\arctan x$)	**R**, or $(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

You may have been able to visualize reflecting the red graph about the line $y = x$ (in brown) in order to obtain the blue graph.



Warning: The line $y = x$ may appear too flat or too steep if the x- and y-axes are scaled differently.

PART D: OTHER INVERSE TRIG FUNCTIONS

The problem with the $\csc^{-1}$, $\sec^{-1}$, and $\cot^{-1}$ functions is that their ranges are not universally agreed upon! (In other words, there is no universal agreement about how the domains of the csc, sec, and cot functions should be restricted.) Different ranges may be used for different purposes.

For example, the range of $\sec^{-1}$ may include angles from Quadrant II (as for $\cos^{-1}$) or from Quadrant III (which tends to be more convenient in Calculus, because tan is positive there).

PART E: REMEMBERING THE RANGES OF INVERSE TRIG FUNCTIONS

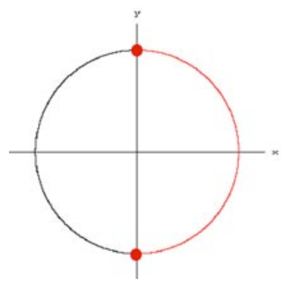
Here are some tricks:

$\sin^{-1}$

Remember that the range of $\sin^{-1}$ is the restricted domain of sin.

We could recall the red graph in Notes 4.73 and find the set of x-coordinates picked up by the graph.

We may also recall the Unit Circle. We want to focus on an arc of the Unit Circle that "picks up" **all** of the possible sin values (corresponding to y-coordinates on the circle) **exactly once** (so that we force the restricted sin function to be one-to-one). We pick the right semicircle, as opposed to the left one.

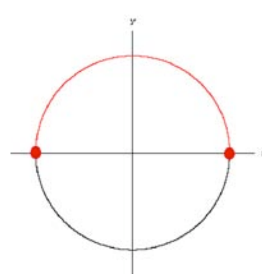


The interval of angles we typically associate with this arc is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note: We may say "angles" when "angle measures" may be more appropriate.

$\cos^{-1}$

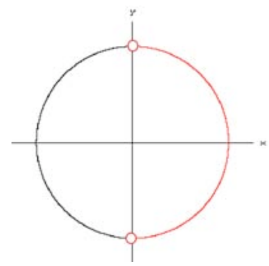
Similarly, we could recall the red graph in Notes 4.75, or we could focus on an arc of the Unit Circle that "picks up" **all** of the possible cos values (corresponding to x-coordinates on the circle) **exactly once**. We pick the top semicircle, as opposed to the bottom one.



The interval of angles we typically associate with this arc is $\left[0, \pi\right]$.

$\tan^{-1}$

Similarly, we could recall the red graph in Notes 4.77, or we could focus on an arc of the Unit Circle that "picks up" **all** of the possible tan values (corresponding to slopes of terminal sides of standard angles) **exactly once**. As for $\sin^{-1}$, we pick the right semicircle, as opposed to the left one, but we must exclude the endpoints, because they correspond to undefined slopes.



The interval we typically associate with this arc is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.