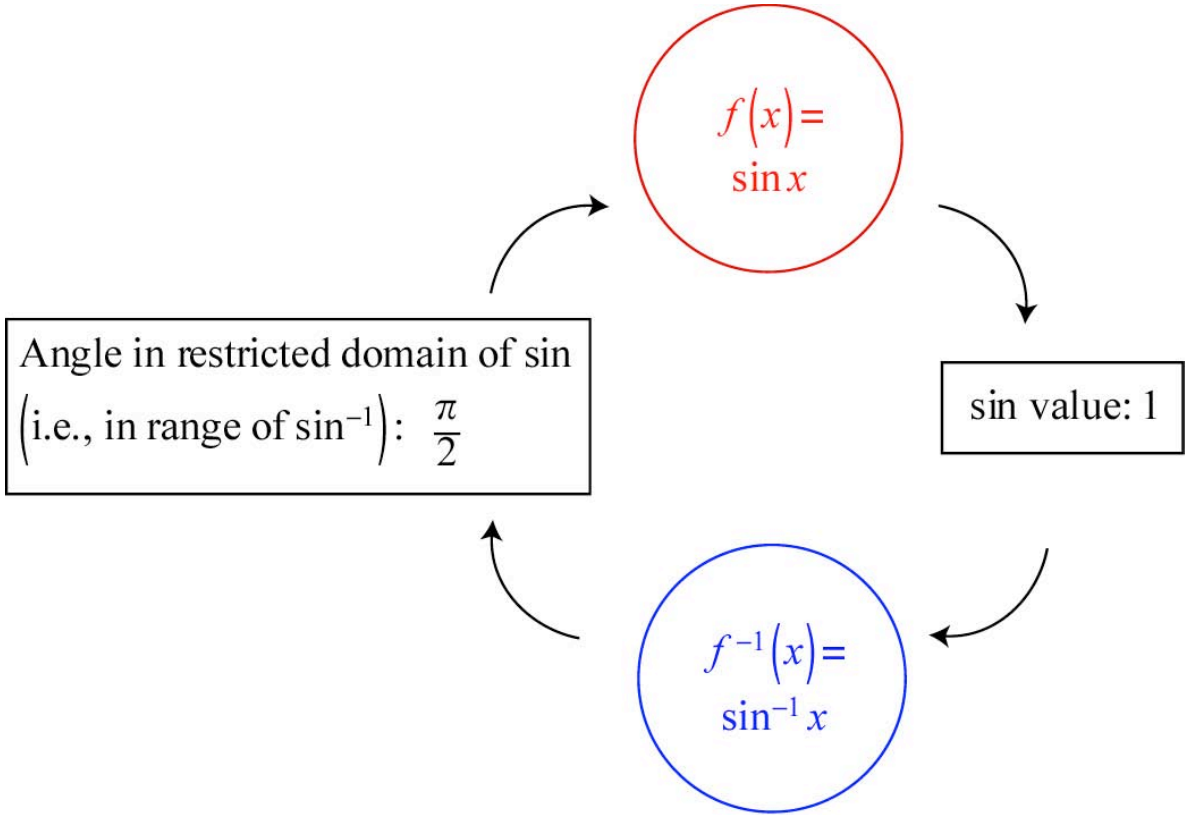
PART F: EVALUATING INVERSE TRIG FUNCTIONS

Think:



A trig function such as sin takes in angles (i.e., real numbers in its domain) as inputs and "spits out" outputs that are trig values (for sin, values between -1 and 1, inclusive).

On the other hand, an inverse trig function such as $\sin^{-1}$ takes in trig values as inputs and "spits out" angles as outputs. These angles must be in the range.

Warning: Although calculators can provide $\sin^{-1}$ values and other inverse trig values using degree measure, it is conventional to use radian measure, instead, since they directly correspond to "real numbers."

Example

Evaluate $\sin^{-1}(1)$, or $\arcsin 1$.

Solution

We know that $\sin\left(\frac{\pi}{2}\right) = 1$.

Although there are other angles whose sine is 1, $\frac{\pi}{2}$ is the only one that is a "legal" output of $\sin^{-1}$, because it is the only one in the range of $\sin^{-1}$, namely $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore, $\sin^{-1}(1) = \boldsymbol{\frac{\pi}{2}}$.

Example

Evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, or $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$.

Solution

What angle in the range of $\sin^{-1}$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, has a sin value of $-\frac{\sqrt{2}}{2}$?

Observe that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, so we would like a brother of $\frac{\pi}{4}$ in Quadrant IV.

Answer: $\boldsymbol{-\frac{\pi}{4}}$.

Observe that the coterminal angle $\frac{7\pi}{4}$, for example, is not in the range.

Example

Evaluate $\sin^{-1}(3)$, or $\arcsin 3$.

Solution

This is **undefined**, because 3 is not a sin value for any angle.
Observe that 3 is not in the domain of $\sin^{-1}$, so it is an "illegal" input.

Example

Evaluate $\cos^{-1}\left(-\frac{1}{2}\right)$, or $\arccos\left(-\frac{1}{2}\right)$.

Solution

What angle in the range of $\cos^{-1}$, $[0, \pi]$, has a cos value of $-\frac{1}{2}$?

Observe that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so we would like a brother of $\frac{\pi}{3}$ in Quadrant II.

Answer: $\mathbf{\frac{2\pi}{3}}$.

Example

$\tan^{-1}(10) \approx 1.47$ [radians]. Observe that $\tan^{-1}(10)$ is **not** undefined, because its domain is **R**; any real number is a slope (or tan value).

PART G: INVERSE PROPERTIES

Inverse Properties: Group 1

If x is an appropriate trig (i.e., sin, cos, tan) value, then:

$$\sin\left(\sin^{-1} x\right) = x \quad \left(\text{if } x \text{ is in } \left[-1,1\right]\right)$$
$$\cos\left(\cos^{-1} x\right) = x \quad \left(\text{if } x \text{ is in } \left[-1,1\right]\right)$$
$$\tan\left(\tan^{-1} x\right) = x \quad \left(\text{if } x \text{ is in } \mathbf{R}\right)$$

Otherwise, we have "**undefined**."

Think: "Unwrapping," or "undoing."

Examples

$$\cos\left(\underbrace{\cos^{-1}\left(0.2\right)}_{\substack{\text{an angle whose} \\ \text{cosine is } 0.2}}\right) = \mathbf{0.2}$$

$$\cos\left(\underbrace{\cos^{-1}\left(10\right)}_{\text{undefined}}\right) \text{ is } \textbf{undefined}$$

$$\tan\left(\underbrace{\tan^{-1}\left(10\right)}_{\substack{\text{an angle whose} \\ \text{tangent is } 10}}\right) = \mathbf{10}$$

Inverse Properties: Group 2

If θ is in the range of the appropriate inverse trig function, then:

$$\sin^{-1}(\sin\theta) = \theta \quad \left(\text{if } \theta \text{ is in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

$$\cos^{-1}(\cos\theta) = \theta \quad \left(\text{if } \theta \text{ is in } [0, \pi]\right)$$

$$\tan^{-1}(\tan\theta) = \theta \quad \left(\text{if } \theta \text{ is in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$$

Example

$$\sin^{-1}\left(\sin\frac{\pi}{10}\right) = \boldsymbol{\frac{\pi}{10}}$$

Think: The $\frac{\pi}{10}$ angle looks over at $\sin^{-1}$ and asks, "Can you spit me out?" The $\sin^{-1}$ function says, "Yes, I can, because you are in my range." Also, sin says, "$\frac{\pi}{10}$ is in my domain, so I'm OK with that."

Note: The "unwrapping" properties described in the box above always work for acute angles θ such as $\frac{\pi}{10}$.

Example

$$\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right)$$
$$= \boldsymbol{\frac{\pi}{6}}$$

"Unwrapping" doesn't work here, because $\frac{5\pi}{6}$ is **not** in the range of $\sin^{-1}$.

The especially talkative $\sin^{-1}$ says to $\frac{5\pi}{6}$, "I can't spit you out, but I can spit out your brother." "When in doubt, work it out," or … observe that we are looking for a brother of $\frac{5\pi}{6}$ that also has a sin value of $\frac{1}{2}$ and that lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Because $\frac{1}{2} > 0$, we look in Quadrant I.

The brother we want is $\frac{\pi}{6}$.

PART H: USING RIGHT TRIANGLES TO WRITE ALGEBRAIC EXPRESSIONS

Example

Write $\cot\left(\cos^{-1}\frac{x}{3}\right)$ as an equivalent algebraic expression in x.

Assume that x is such that all relevant trig and inverse trig values are defined.

Solution

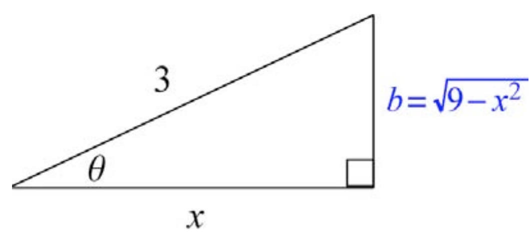
Let the angle $\theta = \cos^{-1}\left(\frac{x}{3}\right)$. Think: $\cot\left(\underbrace{\cos^{-1}\frac{x}{3}}_{=\theta}\right)$

This is true $\Leftrightarrow$ $\cos\theta = \frac{x}{3}$ and θ is in $[0, \pi]$, the range of $\cos^{-1}$.

We may actually assume that θ is acute, without loss of generality.
(If you are dealing with $\csc^{-1}$, $\sec^{-1}$, or $\cot^{-1}$, define the ranges carefully.)

Technical Note: This is a nontrivial observation! See Stewart's Precalculus book for more details.

Construct a model right triangle such that $\cos\theta = \frac{x}{3}$.



Use the Pythagorean Theorem to find an expression for the missing side length, b.

$$x^2 + b^2 = 9$$
$$b^2 = 9 - x^2$$
$$b = \pm\sqrt{9 - x^2}$$
$$b = \sqrt{9 - x^2} \quad \text{(Take the "+" root.)}$$

We see that $\tan\theta = \dfrac{\sqrt{9-x^2}}{x}$.

Warning: $\sqrt{9-x^2} \neq 3-x$.

We then see that $\cot\theta = \dfrac{x}{\sqrt{9-x^2}}$, or $\dfrac{x\sqrt{9-x^2}}{9-x^2}$, which is our answer.

Note: Some books don't require that you rationalize the denominator.

In Calculus: This technique is used when you perform integration using trigonometric substitutions. You will see this in Calculus II: Math 151 at Mesa.