CHAPTER 5: ANALYTIC TRIG

SECTION 5.1: FUNDAMENTAL TRIG IDENTITIES

PART A: WHAT IS AN IDENTITY?

An identity is an equation that is true for all real values of the variable(s) for which all expressions contained within the identity are defined.

For example, $\frac{1}{\left(x^2\right)^3} = \frac{1}{x^6}$ is an identity, because it holds true for all real values of x for which both sides of the equation are defined (i.e., for all real **nonzero** values of x).

If you are given the expression $\frac{1}{\left(x^2\right)^3}$, it may be simplified to form the equivalent expression $\frac{1}{x^6}$.

PART B: LISTS OF FUNDAMENTAL TRIG IDENTITIES

Memorize these in both "directions" (i.e., from left-to-right and from right-to-left).

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sin x = \frac{1}{\csc x}$$

$$\sec x = \frac{1}{\cos x} \qquad \cos x = \frac{1}{\sec x}$$

$$\cot x = \frac{1}{\tan x} \qquad \tan x = \frac{1}{\cot x}$$

Warning: Remember that the reciprocal of $\sin x$ is $\csc x$, not $\sec x$.

Note: We typically treat "0" and "undefined" as reciprocals when we are dealing with trig functions. Your algebra teacher will not want to hear this, though!

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x}$$

Technical Note: See Notes 4.10 on why this is consistent with SOH-CAH-TOA.

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \cot^2 x = \csc^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

See Notes 4.30; see how to derive the last two from the first.

Tip: The squares of $\csc x$ and $\sec x$, which have the "Up-U, Down-U" graphs, are all alone on the right sides of the last two identities. Maybe because they are too profane …. Or maybe because they can never be 0 in value. (Why is that? Look at the left sides.)

Cofunction Identities

If x is measured in radians, then:

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

We have analogous relationships for tan and cot, and for sec and csc; remember that they are sometimes undefined.

Think: Cofunctions of complementary angles are equal.
See Notes 4.15.

Even/Odd (or Negative Angle) Identities

Among the six basic trig functions, cos (and its reciprocal, sec) are even:

$$\cos(-x) = \cos x$$
$$\sec(-x) = \sec x, \text{ when both sides are defined}$$

However, the other four (sin and csc, tan and cot) are odd:

$$\sin(-x) = -\sin x$$
$$\csc(-x) = -\csc x, \text{ when both sides are defined}$$

$$\tan(-x) = -\tan x, \text{ when both sides are defined}$$
$$\cot(-x) = -\cot x, \text{ when both sides are defined}$$

See Notes 4.29.

Note: When an identity is given, it is typically assumed that it holds in all cases for which all expressions contained within it are defined. However, the Reciprocal Identities, for example, can be helpful even when an expression is undefined.

PART C: SIMPLIFYING TRIG EXPRESSIONS

"Simplifying" can mean different things in different settings. For example, is $\csc x$ "simpler" than $\dfrac{1}{\sin x}$? Usually, it is, since $\csc x$ is a more "compact" expression, although the latter can be more useful in other settings.

Each step in a simplification process should be a basic arithmetic or algebraic trick, or it should be an application of a Fundamental Trig Identity (for now). As we learn new identities, our arsenal will grow.

PART D: BREAKING THINGS DOWN INTO sin AND cos

If you are dealing with an expression that contains csc, sec, tan, and/or cot, it may be helpful to write some or all of those irritating expressions in terms of sin and cos. The Reciprocal and Quotient Identities can be especially useful for this purpose.

Example

Simplify $\sin^4 x \cot^4 x$.

Solution

By a Quotient Identity, $\cot x = \dfrac{\cos x}{\sin x}$.

Therefore, $\cot^4 x = (\cot x)^4 = \left(\dfrac{\cos x}{\sin x}\right)^4 = \dfrac{(\cos x)^4}{(\sin x)^4} = \dfrac{\cos^4 x}{\sin^4 x}$.

We usually ignore the middle expressions; they are "automatic" to us.

$$\sin^4 x \cot^4 x = \cancel{\sin^4 x}\left(\frac{\cos^4 x}{\cancel{\sin^4 x}}\right)$$

$$= \mathbf{\cos^4 x}$$

Warning: The given expression simplifies to $\cos^4 x$ **if** x is in the domain of the expression. Don't just assume that the natural (or implied) domain of $\cos^4 x$ is the domain of $\sin^4 x \cot^4 x$. Unless otherwise specified, when we simplify trig expressions, we usually don't state domain restrictions (here, on $\cos^4 x$), although you could argue that that would be more "mathematically proper."

PART E: FACTORING

The old algebraic methodologies still apply.

Example

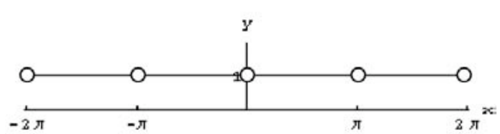
Simplify $\sin^2 x + \sin^2 x \cot^2 x$.

Solution

Factor out the GCF, $\sin^2 x$.

$$\begin{aligned}\sin^2 x + \sin^2 x \cot^2 x &= \left(\sin^2 x\right)\left(1 + \cot^2 x\right) \\ &= \left(\sin^2 x\right)\left(\csc^2 x\right) && \text{(by a Pythagorean Identity)} \\ &= \left(\cancel{\sin^2 x}\right)\left(\frac{1}{\cancel{\sin^2 x}}\right) && \text{(by a Reciprocal Identity)} \\ &= \mathbf{1}\end{aligned}$$

Warning: The given expression simplifies to 1 **if** x is in the domain of the expression. The graph of $y = \sin^2 x + \sin^2 x \cot^2 x$ would resemble the graph of $y = 1$, except that there would be holes (technically, "removable discontinuities") at the values of x that are not in the domain:



Example

Simplify $1-2\cos^2 x+\cos^4 x$.

Solution

We can either factor directly, or we can let $u=\cos x$, rewrite the expression, and then factor.

$$1-2\cos^2 x+\cos^4 x = 1-2u^2+u^4$$

(This is in Quadratic Form; in fact, it is a PST.)

$$=\left(1-u^2\right)^2$$

(Now, go back to x.)

$$=\left(1-\cos^2 x\right)^2$$

(Now, use the basic Pythagorean Identity.)

$$=\left(\sin^2 x\right)^2$$

$$=\mathbf{\sin^4 \boldsymbol{x}}$$

PART F: ADDING AND SUBTRACTING FRACTIONS

Example (#62 in Larson)

Simplify $\dfrac{1}{\sec x+1}-\dfrac{1}{\sec x-1}$.

Solution

The LCD here is the product of the denominators, $(\sec x+1)(\sec x-1)$.

Build up both of the given fractions to obtain this common denominator.

$$\frac{1}{\sec x+1}-\frac{1}{\sec x-1}=\frac{1}{(\sec x+1)}\cdot\frac{(\sec x-1)}{(\sec x-1)}-\frac{1}{(\sec x-1)}\cdot\frac{(\sec x+1)}{(\sec x+1)}$$

Grouping symbols help make things easier to read.
Unite the right side into a single fraction.

$$=\frac{(\sec x-1)-(\sec x+1)}{(\sec x+1)(\sec x-1)}$$

The ( ) around $\sec x+1$ are essential.

$$=\frac{\cancel{\sec x}-1\cancel{-\sec x}-1}{(\sec x+1)(\sec x-1)}$$

$$=\frac{-2}{\sec^2 x-1}$$

We simplified the denominator by using the following multiplication rule from algebra: $(A+B)(A-B)=A^2-B^2$.

The $\sec^2 x$ and the -1 (or $+1$) term should alert you to the possible application of a Pythagorean Identity. In fact,

$$\begin{aligned} 1+\tan^2 x &= \sec^2 x \\ \tan^2 x &= \sec^2 x - 1 \end{aligned}$$

We now have:

$$\begin{aligned} &= \frac{-2}{\tan^2 x} \\ &= -2\left(\frac{1}{\tan^2 x}\right) \\ &= \mathbf{-2\cot^2 x} \end{aligned}$$

You can think of the $\tan^2 x$ as "jumping up" as $\cot^2 x$.

PART G: TRIG SUBSTITUTIONS

In Calculus: This is a key technique of integration. You will see this in Calculus II: Math 151 at Mesa.

Example

Use the trig substitution $x = 4\sin\theta$ to write the algebraic expression $\sqrt{16 - x^2}$ as a trig function of θ, where θ is acute.

Solution

$$\begin{aligned}\sqrt{16 - x^2} &= \sqrt{16 - (4\sin\theta)^2} \\ &= \sqrt{16 - 16\sin^2\theta} \\ &= \sqrt{16(1 - \sin^2\theta)} \\ &= 4\sqrt{(1 - \sin^2\theta)}\end{aligned}$$

Remember the basic Pythagorean Identity:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$

$$= 4\sqrt{\cos^2\theta}$$

Remember that $\sqrt{(blah)^2} = |blah|$.

$$= 4|\cos\theta|$$

Since θ is acute, $\cos\theta > 0$.

$$= \mathbf{4\cos\theta}$$