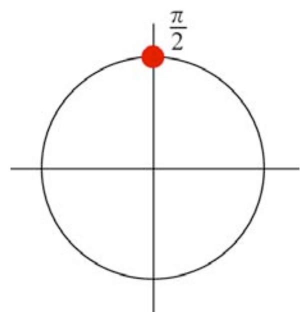
Third factor

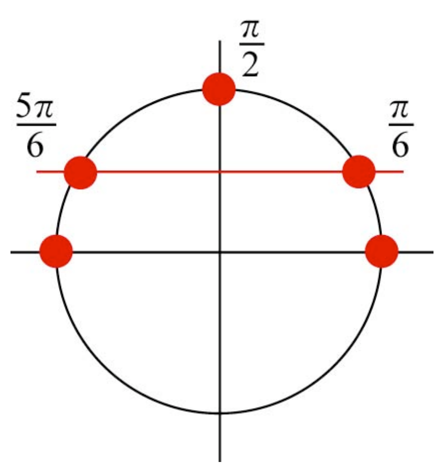
$$u - 1 = 0$$
$$u = 1$$
$$\sin x = 1$$
$$x = \frac{\pi}{2} + 2\pi n \quad (n \text{ integer})$$



Solution set:

$$\left\{ x \,\middle|\, x = \pi n,\ \ x = \frac{\pi}{6} + 2\pi n,\ \ x = \frac{5\pi}{6} + 2\pi n,\ \text{ or }\ x = \frac{\pi}{2} + 2\pi n \quad (n \text{ integer}) \right\}$$

When gathering groups of solutions, you should check to see if there are any more nice symmetries or periodicities you could exploit. No easy ones are apparent here:



PART E: PYTHAGOREAN IDENTITIES

Follow-Up Example

Solve: $2\sin^3 x + \sin x + 3\cos^2 x = 3$

Solution

The $\cos^2 x$ seems like the odd man out, but we can make it look more like the powers of $\sin x$ in the equation. We often prefer conformity.

$$2\sin^3 x + \sin x + 3\cos^2 x = 3$$
$$2\sin^3 x + \sin x + 3\left(1 - \sin^2 x\right) = 3 \quad \text{(by a Pythagorean Identity)}$$
$$2\sin^3 x + \sin x \cancel{+3} - 3\sin^2 x = \cancel{3}^{\,0}$$
$$2\sin^3 x + \sin x - 3\sin^2 x = 0$$
$$2u^3 + u - 3u^2 = 0 \quad (\text{Let } u = \sin x.)$$
$$2u^3 - 3u^2 + u = 0$$

We then proceed as in the previous Example.

PART F: EQUATIONS WITH "MULTIPLE ANGLES"

Example

Solve: $2\sin(4x) = -\sqrt{3}$

Solution

$$2\sin(4x) = -\sqrt{3}$$

Isolate the sin expression.

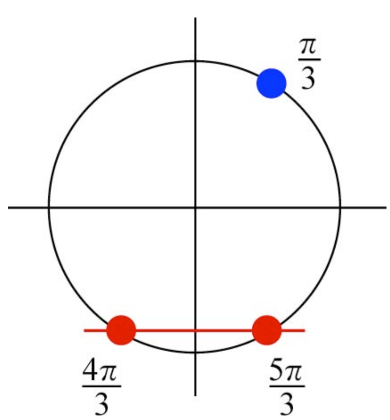
$$\sin\underbrace{(4x)}_{=\theta} = -\frac{\sqrt{3}}{2}$$

Substitution: Let $\theta = 4x$.

$$\sin\theta = -\frac{\sqrt{3}}{2}$$

We will now solve this equation for θ.

Observe that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, so $\frac{\pi}{3}$ will be the reference angle for our solutions for θ. Since $-\frac{\sqrt{3}}{2}$ is a negative sin value, we want brothers of $\frac{\pi}{3}$ in Quadrants III and IV. For multiple angle problems, we may prefer positive brothers, as we will see in our Follow-Up Example.



Our solutions for θ are:

$$\theta = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2\pi n \quad (n \text{ integer})$$

Note: Our solution set will contain the conjunction "or" as an inclusive (not exclusive or limiting) device to gather solutions together. Although the conjunction "and" may have seemed more appropriate in the above phrasing, we will stick with "or" throughout.

From this point on, it is a matter of Algebra.

To find our solutions for x, replace θ with $4x$, and solve for x.

$$4x = \frac{4\pi}{3} + 2\pi n, \quad \text{or} \quad 4x = \frac{5\pi}{3} + 2\pi n \quad (n \text{ integer})$$

$$x = \frac{\frac{4\pi}{3}}{4} + \frac{2\pi}{4} n, \quad \text{or} \quad x = \frac{\frac{5\pi}{3}}{4} + \frac{2\pi}{4} n \quad (n \text{ integer})$$

$$x = \frac{\pi}{3} + \frac{\pi}{2} n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2} n \quad (n \text{ integer})$$

Solution set: $\left\{ x \,\middle|\, x = \frac{\pi}{3} + \frac{\pi}{2} n, \quad \text{or} \quad x = \frac{5\pi}{12} + \frac{\pi}{2} n \quad (n \text{ integer}) \right\}$

Note: The picture for x is considerably more complicated than the picture for θ on the last page. To avoid confusion, you might not want to even think about the picture for x. If you're curious, though, look at the picture at the end of our Follow-Up Example ….

Follow-Up Example

Find all solutions of the equation $2\sin(4x) = -\sqrt{3}$ in the interval $[0, 2\pi)$.
(Surprisingly, this tends to be the much more involved problem!)

Solution

We found the general solution (consisting of two "groups" of solutions) in the previous Example:

$$\left\{ x \,\middle|\, \underbrace{x = \frac{\pi}{3} + \frac{\pi}{2}n}_{\text{Group 1}}, \quad \text{or} \quad \underbrace{x = \frac{5\pi}{12} + \frac{\pi}{2}n}_{\text{Group 2}} \quad (n \text{ integer}) \right\}$$

We could plug in integer values for n and evaluate to obtain particular solutions.

Instead, let's try a more efficient approach using the "Increment" idea we used for trig graphs in Chapter 4. Observe that, every time we increase n by 1, that has the effect of adding $\frac{\pi}{2}$ in both groups of solutions.
(Think: Distributive Property of Multiplication over Addition.)
We can think of $\frac{\pi}{2}$ as our "Increment" in both groups.

<u>Group 1</u>: $x = \frac{\pi}{3} + \frac{\pi}{2} n \quad (n \text{ integer})$

Observe that 6 is the LCD.

Start with: $\frac{\pi}{3}$, which equals $\frac{2\pi}{6}$

Increment: $\frac{\pi}{2}$, which equals $\frac{3\pi}{6}$

The simplified solutions are in bold below.

(Optional)	Particular Solutions
$n = 0$	$\boldsymbol{\frac{\pi}{3}} = \frac{2\pi}{6}$
$n = 1$	$\boldsymbol{\frac{5\pi}{6}}$
$n = 2$	$\frac{8\pi}{6} = \boldsymbol{\frac{4\pi}{3}}$
$n = 3$	$\boldsymbol{\frac{11\pi}{6}}$

Observe that, if the integer $n < 0$, the resulting value is negative, so it cannot be included in the solution set. Remember that we are only looking for solutions in the interval $[0, 2\pi)$. The fact that the $n = 0$ solution "works" here is a result of our preference for the smallest positive brothers that "worked" back in Notes 5.34.

Observe that, if the integer $n > 3$, the resulting value is at least 2π, so it cannot be included, either.

This all works smoothly because $\frac{\pi}{3}$ is the smallest positive solution (angle) in this group. If you had started with another angle in the group, make sure that you add and/or subtract the Increment in such a way that you "sweep through" the interval $[0, 2\pi)$ and pick up all real solutions in there.

Group 2: $x = \frac{5\pi}{12} + \frac{\pi}{2}n \quad (n \text{ integer})$

Observe that 12 is the LCD.

Start with: $\frac{5\pi}{12}$

Increment: $\frac{\pi}{2}$, which equals $\frac{6\pi}{12}$

The simplified solutions are in bold below.

(Optional)	Particular Solutions
$n = 0$	$\boldsymbol{\frac{5\pi}{12}}$
$n = 1$	$\boldsymbol{\frac{11\pi}{12}}$
$n = 2$	$\boldsymbol{\frac{17\pi}{12}}$
$n = 3$	$\boldsymbol{\frac{23\pi}{12}}$

Observe that, if the integer $n < 0$, the resulting value is negative, so it cannot be included in the solution set.

Observe that, if the integer $n > 3$, the resulting value is at least 2π, so it cannot be included, either.

Solution set:

(Instructors typically do not require numbers in solution sets to be written in increasing order. Sets are typically assumed to be unordered, anyway.)

There are 8 solutions in the interval $[0, 2\pi)$:

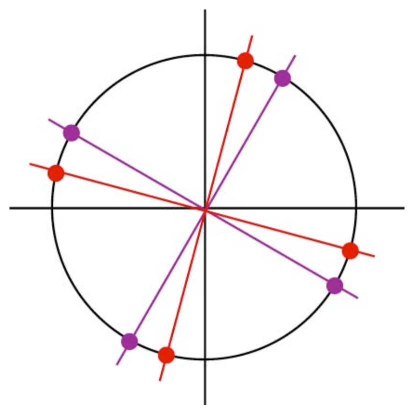
$$\left\{ \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12} \right\}$$

Idea:

The two "spider eggs" at the red points of interest for the θ figure in Notes 5.34 each break open and produce four spiders placed evenly (periodically) along the Unit Circle.

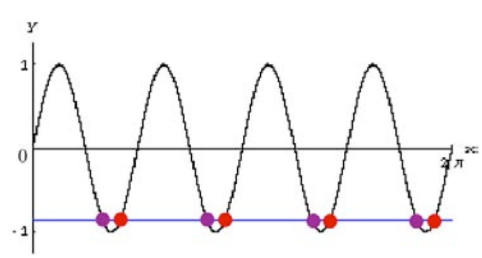
Basically, we take the $\frac{4\pi}{3}$ angle and three of its coterminal "twins" and the $\frac{5\pi}{3}$ angle and three of its coterminal "twins" and divide each of them by 4.

The first group corresponds to the purple points below.
The second group corresponds to the red points below.



These points correspond to the intersection points of the graphs of $y=\sin(4x)$ (in black) and $y=-\frac{\sqrt{3}}{2}$ (in blue) below.

Their x-coordinates are solutions to $\sin(4x)=-\frac{\sqrt{3}}{2}$.



PART G: USING INVERSE FUNCTIONS

Example

Solve: $3\sin x - 1 = 0$. (Find all real solutions.)

Solution

$$3\sin x - 1 = 0$$

Isolate the sin expression.

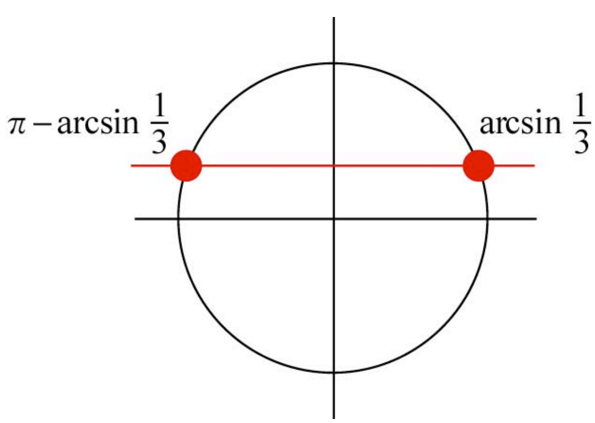
$$\sin x = \frac{1}{3}$$

$\frac{1}{3}$ is not a special sin value, although it does lie in the range of the $\sin x$ function, $\left[-1, 1\right]$. This equation has real solutions, but we need to use "arcsin" or "$\sin^{-1}$" notation to express them exactly.

Since the sin value $\frac{1}{3} > 0$, $\arcsin\left(\frac{1}{3}\right)$ represents an acute angle in Quadrant I, and we also want a brother in Quadrant II.

Remember that radians are the assumed measure for angles.

Note: $\arcsin\left(\frac{1}{3}\right)$ is about 19.5°. Its brother below is about 160.5°.



The two particular solutions in $\left[0, 2\pi\right)$ are $\arcsin\left(\frac{1}{3}\right)$ and $\pi - \arcsin\left(\frac{1}{3}\right)$.

Considering all coterminal "twin" angles, the general solution set is:

$$\left\{ x \,\middle|\, x = \arcsin\left(\frac{1}{3}\right) + 2\pi n, \quad \text{or} \quad x = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi n \quad \left(n \text{ integer}\right) \right\}$$

… or, using $\sin^{-1}$ notation, …

$$\left\{ x \,\middle|\, x = \sin^{-1}\left(\frac{1}{3}\right) + 2\pi n, \quad \text{or} \quad x = \pi - \sin^{-1}\left(\frac{1}{3}\right) + 2\pi n \quad \left(n \text{ integer}\right) \right\}$$