

HANDOUT ON SECTIONS 5.4 AND 5.5: MORE TRIG IDENTITIES – MEMORIZE!

SUM IDENTITIES

Memorize:

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

Think: “Sum of the mixed-up products”
(Multiplication and addition are commutative, but start with the $\sin u \cos v$ term in anticipation of the Difference Identities.)

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

Think: “Cosines [product] – Sines [product]”

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

Think: " $\frac{\text{Sum}}{1 - \text{Product}}$ "

DIFFERENCE IDENTITIES

Memorize:

Simply take the Sum Identities above and change every sign in sight!

$$\sin(u - v) = \sin u \cos v - \cos u \sin v$$

(Make sure that the right side of your identity for $\sin(u + v)$ started with the $\sin u \cos v$ term!)

$$\cos(u - v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Obtaining the Difference Identities from the Sum Identities:

Replace v with $(-v)$ and use the fact that \sin and \tan are odd, while \cos is even.

For example,

$$\begin{aligned}\sin(u - v) &= \sin[u + (-v)] \\ &= \sin u \cos(-v) + \cos u \sin(-v) \\ &= \sin u \cos v - \cos u \sin v\end{aligned}$$

DOUBLE-ANGLE (Think: Angle-Reducing, if $u > 0$) IDENTITIES

Memorize:

(Also be prepared to recognize and know these “right-to-left”)

$$\sin(2u) = 2 \sin u \cos u$$

Think: “Twice the product”

Reading “right-to-left,” we have:

$$2 \sin u \cos u = \sin(2u)$$

(This is helpful when simplifying.)

$$\cos(2u) = \cos^2 u - \sin^2 u$$

Think: “Cosines – Sines” (again)

Reading “right-to-left,” we have:

$$\cos^2 u - \sin^2 u = \cos(2u)$$

Contrast this with the Pythagorean Identity:

$$\cos^2 u + \sin^2 u = 1$$

$$\tan(2u) = \frac{2 \tan u}{1 - \tan^2 u}$$

(Hard to memorize; we’ll show how to obtain it.)

Notice that these identities are “angle-reducing” (if $u > 0$) in that they allow you to go from trig functions of $(2u)$ to trig functions of simply u .

Obtaining the Double-Angle Identities from the Sum Identities:

Take the Sum Identities, replace v with u , and simplify.

$$\begin{aligned}\sin(2u) &= \sin(u + u) \\ &= \sin u \cos u + \cos u \sin u \quad (\text{From Sum Identity}) \\ &= \sin u \cos u + \sin u \cos u \quad (\text{Like terms!!}) \\ &= 2 \sin u \cos u\end{aligned}$$

$$\begin{aligned}\cos(2u) &= \cos(u + u) \\ &= \cos u \cos u - \sin u \sin u \quad (\text{From Sum Identity}) \\ &= \cos^2 u - \sin^2 u\end{aligned}$$

$$\begin{aligned}\tan(2u) &= \tan(u + u) \\ &= \frac{\tan u + \tan u}{1 - \tan u \tan u} \quad (\text{From Sum Identity}) \\ &= \frac{2 \tan u}{1 - \tan^2 u}\end{aligned}$$

This is a “last resort” if you forget the Double-Angle Identities, but you will need to recall the Double-Angle Identities quickly!

One possible exception: Since the $\tan(2u)$ identity is harder to remember, you may prefer to remember the Sum Identity for $\tan(u + v)$ and then derive the $\tan(2u)$ identity this way.

If you’re quick with algebra, you may prefer to go in reverse: memorize the Double-Angle Identities, and then guess the Sum Identities.

Memorize These Three Versions of the Double-Angle Identity for $\cos(2u)$:

Let's begin with the version we've already seen:

$$\text{Version 1: } \cos(2u) = \cos^2 u - \sin^2 u$$

Also know these two, from "left-to-right," and from "right-to-left":

$$\text{Version 2: } \cos(2u) = 1 - 2 \sin^2 u$$

$$\text{Version 3: } \cos(2u) = 2 \cos^2 u - 1$$

Obtaining Versions 2 and 3 from Version 1

It's tricky to remember Versions 2 and 3, but you can obtain them from Version 1 by using the Pythagorean Identity $\sin^2 u + \cos^2 u = 1$ written in different ways.

To obtain Version 2, which contains $\sin^2 u$, we replace $\cos^2 u$ with $(1 - \sin^2 u)$.

$$\begin{aligned} \cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\ &= \underbrace{(1 - \sin^2 u)}_{\substack{\text{from Pythagorean} \\ \text{Identity}}} - \sin^2 u \\ &= 1 - \sin^2 u - \sin^2 u \\ &= 1 - 2 \sin^2 u && (\Rightarrow \text{Version 2}) \end{aligned}$$

To obtain Version 3, which contains $\cos^2 u$, we replace $\sin^2 u$ with $(1 - \cos^2 u)$.

$$\begin{aligned} \cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\ &= \cos^2 u - \underbrace{(1 - \cos^2 u)}_{\substack{\text{from Pythagorean} \\ \text{Identity}}} \\ &= \cos^2 u - 1 + \cos^2 u \\ &= 2 \cos^2 u - 1 && (\Rightarrow \text{Version 3}) \end{aligned}$$

POWER-REDUCING IDENTITIES (“PRIs”)

(These are called the “Half-Angle Formulas” in some books.)

Memorize:

Then,

$$\sin^2 u = \frac{1 - \cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} - \frac{1}{2}\cos(2u) \qquad \tan^2 u = \frac{\sin^2 u}{\cos^2 u} = \frac{1 - \cos(2u)}{1 + \cos(2u)}$$

$$\cos^2 u = \frac{1 + \cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} + \frac{1}{2}\cos(2u)$$

Actually, you just need to memorize one of the $\sin^2 u$ or $\cos^2 u$ identities and then switch the visible sign to get the other. Think: “sin” is “bad” or “negative”; this is a reminder that the minus sign belongs in the $\sin^2 u$ formula.

Obtaining the Power-Reducing Identities from the Double-Angle Identities for $\cos(2u)$

To obtain the identity for $\sin^2 u$, start with Version 2 of the $\cos(2u)$ identity:

$$\cos(2u) = 1 - 2 \sin^2 u$$

Now, solve for $\sin^2 u$.

$$2 \sin^2 u = 1 - \cos(2u)$$

$$\sin^2 u = \frac{1 - \cos(2u)}{2}$$

To obtain the identity for $\cos^2 u$, start with Version 3 of the $\cos(2u)$ identity:

$$\cos(2u) = 2 \cos^2 u - 1$$

Now, switch sides and solve for $\cos^2 u$.

$$2 \cos^2 u - 1 = \cos(2u)$$

$$2 \cos^2 u = 1 + \cos(2u)$$

$$\cos^2 u = \frac{1 + \cos(2u)}{2}$$

HALF-ANGLE IDENTITIES

Instead of memorizing these outright, it may be easier to derive them from the Power-Reducing Identities (PRIs). We use the substitution $\theta = 2u$. (See **Obtaining ...** below.)

The Identities:

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$$

For a given θ , the choices among the \pm signs depend on the Quadrant that $\frac{\theta}{2}$ lies in.

Here, the \pm symbols indicate incomplete knowledge; unlike when we deal with the Quadratic Formula, we do not take both signs for any of the above formulas for a given

θ . There are no \pm symbols in the last two $\tan\left(\frac{\theta}{2}\right)$ formulas; there is no problem there of incomplete knowledge regarding signs.

One way to remember the last two $\tan\left(\frac{\theta}{2}\right)$ formulas: Keep either the numerator or the denominator of the radicand of the first formula, stick $\sin\theta$ in the other part of the fraction, and remove the radical sign and the \pm symbol.

Obtaining the Half-Angle Identities from the Power-Reducing Identities (PRIs):

For the $\sin\left(\frac{\theta}{2}\right)$ identity, we begin with the PRI:

$$\sin^2 u = \frac{1 - \cos(2u)}{2}$$

$$\text{Let } u = \frac{\theta}{2}, \text{ or } \theta = 2u.$$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{2}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos\theta}{2}} \quad (\text{by the Square Root Method})$$

Again, the choice among the \pm signs depends on the Quadrant that $\frac{\theta}{2}$ lies in.

The story is similar for the $\cos\left(\frac{\theta}{2}\right)$ and the $\tan\left(\frac{\theta}{2}\right)$ identities.

What about the last two formulas for $\tan\left(\frac{\theta}{2}\right)$? The key trick is multiplication by trig conjugates. For example:

$$\begin{aligned} \tan\left(\frac{\theta}{2}\right) &= \pm \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} \\ &= \pm \sqrt{\frac{(1 - \cos\theta)(1 - \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)}} \\ &= \pm \sqrt{\frac{(1 - \cos\theta)^2}{1 - \cos^2\theta}} \\ &= \pm \sqrt{\frac{(1 - \cos\theta)^2}{\sin^2\theta}} \\ &= \pm \sqrt{\left(\frac{1 - \cos\theta}{\sin\theta}\right)^2} \\ &= \pm \left| \frac{1 - \cos\theta}{\sin\theta} \right| \quad \left(\text{because } \sqrt{blah^2} = |blah| \right) \end{aligned}$$

Now, $1 - \cos \theta \geq 0$ for all real θ , and $\tan\left(\frac{\theta}{2}\right)$ has the same sign as $\sin \theta$ (can you see why?), so ...

$$= \frac{1 - \cos \theta}{\sin \theta}$$

To get the third formula, use the numerator's (instead of the denominator's) trig conjugate, $1 + \cos \theta$, when multiplying into the numerator and the denominator of the radicand in the first few steps.

PRODUCT-TO-SUM IDENTITIES (Given as necessary on exams)

These can be verified from right-to-left using the Sum and Difference Identities.

The Identities:

$$\sin u \sin v = \frac{1}{2} [\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2} [\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2} [\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2} [\sin(u + v) - \sin(u - v)]$$

SUM-TO-PRODUCT IDENTITIES (Given as necessary on exams)

(See textbook for proofs.)

The Identities:

$$\sin x + \sin y = 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$

$$\sin x - \sin y = 2 \cos\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$