HANDOUT ON SECTIONS 5.4 AND 5.5: MORE TRIG IDENTITIES – MEMORIZE!

SUM IDENTITIES

Memorize:

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

Think: "Sum of the mixed-up products"
(Multiplication and addition are commutative, but start with the $\sin u \cos v$ term in anticipation of the Difference Identities.)

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

Think: "Cosines [product] – Sines [product]"

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

Think: $"\dfrac{\text{Sum}}{1-\text{Product}}"$

DIFFERENCE IDENTITIES

Memorize:

Simply take the Sum Identities above and change every sign in sight!

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

(Make sure that the right side of your identity for $\sin(u+v)$ started with the $\sin u \cos v$ term!)

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Obtaining the Difference Identities from the Sum Identities:

Replace v with $(-v)$ and use the fact that sin and tan are odd, while cos is even.

For example,

$$\begin{aligned}\sin(u-v) &= \sin[u+(-v)] \\ &= \sin u \cos(-v) + \cos u \sin(-v) \\ &= \sin u \cos v - \cos u \sin v\end{aligned}$$

DOUBLE-ANGLE (Think: Angle-Reducing, if $u > 0$) IDENTITIES

Memorize:

(Also be prepared to recognize and know these "right-to-left")

$$\sin(2u) = 2\sin u \cos u$$

Think: "Twice the product"

Reading "right-to-left," we have:

$$2\sin u \cos u = \sin(2u)$$

(This is helpful when simplifying.)

$$\cos(2u) = \cos^2 u - \sin^2 u$$

Think: "Cosines – Sines" (again)

Reading "right-to-left," we have:

$$\cos^2 u - \sin^2 u = \cos(2u)$$

Contrast this with the Pythagorean Identity:

$$\cos^2 u + \sin^2 u = 1$$

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

(Hard to memorize; we'll show how to obtain it.)

Notice that these identities are "angle-reducing" (if $u > 0$) in that they allow you to go from trig functions of $(2u)$ to trig functions of simply u.

Obtaining the Double-Angle Identities from the Sum Identities:

Take the Sum Identities, replace v with u, and simplify.

$$\begin{aligned}\sin(2u) &= \sin(u+u)\\ &= \sin u \cos u + \cos u \sin u \quad \text{(From Sum Identity)}\\ &= \sin u \cos u + \sin u \cos u \quad \text{(Like terms!!)}\\ &= 2 \sin u \cos u\end{aligned}$$

$$\begin{aligned}\cos(2u) &= \cos(u+u)\\ &= \cos u \cos u - \sin u \sin u \quad \text{(From Sum Identity)}\\ &= \cos^2 u - \sin^2 u\end{aligned}$$

$$\begin{aligned}\tan(2u) &= \tan(u+u)\\ &= \frac{\tan u + \tan u}{1 - \tan u \tan u} \quad \text{(From Sum Identity)}\\ &= \frac{2 \tan u}{1 - \tan^2 u}\end{aligned}$$

This is a "last resort" if you forget the Double-Angle Identities, but you will need to recall the Double-Angle Identities quickly!

One possible exception: Since the $\tan(2u)$ identity is harder to remember, you may prefer to remember the Sum Identity for $\tan(u+v)$ and then derive the $\tan(2u)$ identity this way.

If you're quick with algebra, you may prefer to go in reverse: memorize the Double-Angle Identities, and then guess the Sum Identities.

Memorize These Three Versions of the Double-Angle Identity for $\cos(2u)$:

Let's begin with the version we've already seen:

Version 1: $\cos(2u) = \cos^2 u - \sin^2 u$

Also know these two, from "left-to-right," and from "right-to-left":

Version 2: $\cos(2u) = 1 - 2\sin^2 u$

Version 3: $\cos(2u) = 2\cos^2 u - 1$

Obtaining Versions 2 and 3 from Version 1

It's tricky to remember Versions 2 and 3, but you can obtain them from Version 1 by using the Pythagorean Identity $\sin^2 u + \cos^2 u = 1$ written in different ways.

To obtain Version 2, which contains $\sin^2 u$, we replace $\cos^2 u$ with $\left(1 - \sin^2 u\right)$.

$$\begin{aligned}
\cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\
&= \underbrace{\left(1 - \sin^2 u\right)}_{\text{from Pythagorean Identity}} - \sin^2 u \\
&= 1 - \sin^2 u - \sin^2 u \\
&= 1 - 2\sin^2 u && (\Rightarrow \text{Version 2})
\end{aligned}$$

To obtain Version 3, which contains $\cos^2 u$, we replace $\sin^2 u$ with $\left(1 - \cos^2 u\right)$.

$$\begin{aligned}
\cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\
&= \cos^2 u - \underbrace{\left(1 - \cos^2 u\right)}_{\text{from Pythagorean Identity}} \\
&= \cos^2 u - 1 + \cos^2 u \\
&= 2\cos^2 u - 1 && (\Rightarrow \text{Version 3})
\end{aligned}$$

POWER-REDUCING IDENTITIES ("PRIs")

(These are called the "Half-Angle Formulas" in some books.)

Memorize:

$$\sin^2 u = \frac{1-\cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} - \frac{1}{2}\cos(2u)$$

$$\cos^2 u = \frac{1+\cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} + \frac{1}{2}\cos(2u)$$

Then,

$$\tan^2 u = \frac{\sin^2 u}{\cos^2 u} = \frac{1-\cos(2u)}{1+\cos(2u)}$$

Actually, you just need to memorize one of the $\sin^2 u$ or $\cos^2 u$ identities and then switch the visible sign to get the other. Think: "sin" is "bad" or "negative"; this is a reminder that the minus sign belongs in the $\sin^2 u$ formula.

Obtaining the Power-Reducing Identities from the Double-Angle Identities for $\cos(2u)$

To obtain the identity for $\sin^2 u$, start with Version 2 of the $\cos(2u)$ identity:

$$\cos(2u) = 1 - 2\sin^2 u$$

Now, solve for $\sin^2 u$.

$$2\sin^2 u = 1 - \cos(2u)$$

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

To obtain the identity for $\cos^2 u$, start with Version 3 of the $\cos(2u)$ identity:

$$\cos(2u) = 2\cos^2 u - 1$$

Now, switch sides and solve for $\cos^2 u$.

$$2\cos^2 u - 1 = \cos(2u)$$

$$2\cos^2 u = 1 + \cos(2u)$$

$$\cos^2 u = \frac{1+\cos(2u)}{2}$$

HALF-ANGLE IDENTITIES

Instead of memorizing these outright, it may be easier to derive them from the Power-Reducing Identities (PRIs). We use the substitution $\theta = 2u$. (See **Obtaining …** below.)

The Identities:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

For a given θ, the choices among the $\pm$ signs depend on the Quadrant that $\frac{\theta}{2}$ lies in. Here, the $\pm$ symbols indicate incomplete knowledge; unlike when we deal with the Quadratic Formula, we do not take both signs for any of the above formulas for a given θ. There are no $\pm$ symbols in the last two $\tan\left(\frac{\theta}{2}\right)$ formulas; there is no problem there of incomplete knowledge regarding signs.

One way to remember the last two $\tan\left(\frac{\theta}{2}\right)$ formulas: Keep either the numerator or the denominator of the radicand of the first formula, stick $\sin\theta$ in the other part of the fraction, and remove the radical sign and the $\pm$ symbol.

Obtaining the Half-Angle Identities from the Power-Reducing Identities (PRIs):

For the $\sin\left(\frac{\theta}{2}\right)$ identity, we begin with the PRI:

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

Let $u = \frac{\theta}{2}$, or $\theta = 2u$.

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \text{(by the Square Root Method)}$$

Again, the choice among the $\pm$ signs depends on the Quadrant that $\frac{\theta}{2}$ lies in.

The story is similar for the $\cos\left(\frac{\theta}{2}\right)$ and the $\tan\left(\frac{\theta}{2}\right)$ identities.

What about the last two formulas for $\tan\left(\frac{\theta}{2}\right)$? The key trick is multiplication by trig conjugates. For example:

$$\begin{aligned}
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}\cdot\frac{(1-\cos\theta)}{(1-\cos\theta)}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} \\
&= \pm\sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \\
&= \pm\sqrt{\left(\frac{1-\cos\theta}{\sin\theta}\right)^2} \\
&= \pm\left|\frac{1-\cos\theta}{\sin\theta}\right| \qquad \left(\text{because } \sqrt{blah^2} = |blah|\right)
\end{aligned}$$

Now, $1-\cos\theta \geq 0$ for all real θ, and $\tan\left(\frac{\theta}{2}\right)$ has the same sign as $\sin\theta$ (can you see why?), so …

$$= \frac{1-\cos\theta}{\sin\theta}$$

To get the third formula, use the numerator's (instead of the denominator's) trig conjugate, $1+\cos\theta$, when multiplying into the numerator and the denominator of the radicand in the first few steps.

PRODUCT-TO-SUM IDENTITIES (Given as necessary on exams)

These can be verified from right-to-left using the Sum and Difference Identities.

The Identities:

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$

$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$

$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$

$$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

SUM-TO-PRODUCT IDENTITIES (Given as necessary on exams)

(See textbook for proofs.)

The Identities:

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$