

## MORE TRIG IDENTITIES – MEMORIZE!

### SUM IDENTITIES

**Memorize:**

**#1**  $\sin(u + v) = \sin(u)\cos(v) + \cos(u)\sin(v)$  ( ) on right side are suggested.

Think: “Sum of the mixed-up products”

(Multiplication and addition are commutative, but start with the  $\sin u \cos v$  term in anticipation of the Difference Identities.)

**#2**  $\cos(u + v) = \cos u \cos v - \sin u \sin v$

Think: “Cosines [product] – Sines [product]”

**#3**  $\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$

Think: “ $\frac{\text{Sum}}{1 - \text{Product}}$ ”

### DIFFERENCE IDENTITIES

**Memorize:**

Simply take the Sum Identities above and change every sign in sight!

**#4**  $\sin(u - v) = \sin u \cos v - \cos u \sin v$

(Make sure that the right side of your identity for  $\sin(u + v)$  started with the  $\sin u \cos v$  term!)

**#5**  $\cos(u - v) = \cos u \cos v + \sin u \sin v$

**#6**  $\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$

**Obtaining the Difference Identities from the Sum Identities:**

Replace  $v$  with  $(-v)$  and use the fact that  $\sin$  and  $\tan$  are odd, while  $\cos$  is even.

For example,

$$\begin{aligned}\sin(u - v) &= \sin[u + (-v)] \\ &= \sin u \cos(-v) + \cos u \sin(-v) \\ &= \sin u \cos v - \cos u \sin v\end{aligned}$$

## DOUBLE-ANGLE (Think: Angle-Reducing, if $u > 0$ ) IDENTITIES

**Memorize:**

(Also be prepared to recognize and know these “right-to-left”)

**#7**  $\sin(2u) = 2 \sin u \cos u$

Think: “Twice the product”

Reading “right-to-left,” we have:

$$2 \sin u \cos u = \sin(2u)$$

(This is helpful when simplifying.)

**#8**  $\cos(2u) = \cos^2 u - \sin^2 u$

Think: “Cosines – Sines” (again)

Reading “right-to-left,” we have:

$$\cos^2 u - \sin^2 u = \cos(2u)$$

Contrast this with the Pythagorean Identity:

$$\cos^2 u + \sin^2 u = 1$$

**#9**  $\tan(2u) = \frac{2 \tan u}{1 - \tan^2 u}$

(Hard to memorize; we’ll show how to obtain it.)

Notice that these identities are “angle-reducing” (if  $u > 0$ ) in that they allow you to go from trig functions of  $(2u)$  to trig functions of simply  $u$ .

## Obtaining the Double-Angle Identities from the Sum Identities:

Take the Sum Identities, replace  $v$  with  $u$ , and simplify.

$$\begin{aligned}\sin(2u) &= \sin(u + u) \\ &= \sin u \cos u + \cos u \sin u \quad (\text{From Sum Identity}) \\ &= \sin u \cos u + \sin u \cos u \quad (\text{Like terms!!}) \\ &= 2 \sin u \cos u\end{aligned}$$

$$\begin{aligned}\cos(2u) &= \cos(u + u) \\ &= \cos u \cos u - \sin u \sin u \quad (\text{From Sum Identity}) \\ &= \cos^2 u - \sin^2 u\end{aligned}$$

$$\begin{aligned}\tan(2u) &= \tan(u + u) \\ &= \frac{\tan u + \tan u}{1 - \tan u \tan u} \quad (\text{From Sum Identity}) \\ &= \frac{2 \tan u}{1 - \tan^2 u}\end{aligned}$$

This is a “last resort” if you forget the Double-Angle Identities, but you will need to recall the Double-Angle Identities quickly!

One possible exception: Since the  $\tan(2u)$  identity is harder to remember, you may prefer to remember the Sum Identity for  $\tan(u + v)$  and then derive the  $\tan(2u)$  identity this way.

If you’re quick with algebra, you may prefer to go in reverse: memorize the Double-Angle Identities, and then guess the Sum Identities.

## Memorize These Three Versions of the Double-Angle Identity for $\cos(2u)$ :

Let's begin with the version we've already seen:

**#8**            Version 1:     $\cos(2u) = \cos^2 u - \sin^2 u$

Also know these two, from “left-to-right,” and from “right-to-left”:

**#10**            Version 2:     $\cos(2u) = 1 - 2 \sin^2 u$

**#11**            Version 3:     $\cos(2u) = 2 \cos^2 u - 1$

### Obtaining Versions 2 and 3 from Version 1

It's tricky to remember Versions 2 and 3, but you can obtain them from Version 1 by using the Pythagorean Identity  $\sin^2 u + \cos^2 u = 1$  written in different ways.

To obtain Version 2, which contains  $\sin^2 u$ , we replace  $\cos^2 u$  with  $(1 - \sin^2 u)$ .

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\ &= \underbrace{(1 - \sin^2 u)}_{\substack{\text{from Pythagorean} \\ \text{Identity}}} - \sin^2 u \\ &= 1 - \sin^2 u - \sin^2 u \\ &= 1 - 2 \sin^2 u && (\Rightarrow \text{Version 2})\end{aligned}$$

To obtain Version 3, which contains  $\cos^2 u$ , we replace  $\sin^2 u$  with  $(1 - \cos^2 u)$ .

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u && \text{(Version 1)} \\ &= \cos^2 u - \underbrace{(1 - \cos^2 u)}_{\substack{\text{from Pythagorean} \\ \text{Identity}}} \\ &= \cos^2 u - 1 + \cos^2 u \\ &= 2 \cos^2 u - 1 && (\Rightarrow \text{Version 3})\end{aligned}$$

## POWER-REDUCING IDENTITIES (“PRIs”)

(These are called the “Half-Angle Formulas” in some books.)

**Memorize:**

**Then,**

**#12**

$$\sin^2 u = \frac{1 - \cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} - \frac{1}{2}\cos(2u)$$

**#14**

$$\tan^2 u = \frac{\sin^2 u}{\cos^2 u} = \frac{1 - \cos(2u)}{1 + \cos(2u)}$$

**#13**

$$\cos^2 u = \frac{1 + \cos(2u)}{2} \quad \text{or} \quad \frac{1}{2} + \frac{1}{2}\cos(2u)$$

Actually, you just need to memorize one of the  $\sin^2 u$  or  $\cos^2 u$  identities and then switch the visible sign to get the other. Think: “sin” is “bad” or “negative”; this is a reminder that the minus sign belongs in the  $\sin^2 u$  formula.

### **Obtaining the Power-Reducing Identities from the Double-Angle Identities for $\cos(2u)$**

To obtain the identity for  $\sin^2 u$ , start with Version 2 of the  $\cos(2u)$  identity:

$$\cos(2u) = 1 - 2 \sin^2 u$$

Now, solve for  $\sin^2 u$ .

$$2 \sin^2 u = 1 - \cos(2u)$$

$$\sin^2 u = \frac{1 - \cos(2u)}{2}$$

To obtain the identity for  $\cos^2 u$ , start with Version 3 of the  $\cos(2u)$  identity:

$$\cos(2u) = 2 \cos^2 u - 1$$

Now, switch sides and solve for  $\cos^2 u$ .

$$2 \cos^2 u - 1 = \cos(2u)$$

$$2 \cos^2 u = 1 + \cos(2u)$$

$$\cos^2 u = \frac{1 + \cos(2u)}{2}$$

## HALF-ANGLE IDENTITIES

Instead of memorizing these outright, it may be easier to derive them from the Power-Reducing Identities (PRIs). We use the substitution  $\theta = 2u$ . (See **Obtaining ...** below.)

### The Identities:

$$\#15 \quad \sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos\theta}{2}}$$

$$\#16 \quad \cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 + \cos\theta}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta}$$



For a given  $\theta$ , the choices among the  $\pm$  signs depend on the Quadrant that  $\frac{\theta}{2}$  lies in.

Here, the  $\pm$  symbols indicate incomplete knowledge; unlike when we deal with the Quadratic Formula, we do not take both signs for any of the above formulas for a given  $\theta$ . There are no  $\pm$  symbols in the last two  $\tan\left(\frac{\theta}{2}\right)$  formulas; there is no problem there of incomplete knowledge regarding signs.

One way to remember the last two  $\tan\left(\frac{\theta}{2}\right)$  formulas: Keep either the numerator or the denominator of the radicand of the first formula, stick  $\sin\theta$  in the other part of the fraction, and remove the radical sign and the  $\pm$  symbol.

**Obtaining the Half-Angle Identities from the Power-Reducing Identities (PRIs):**

For the  $\sin\left(\frac{\theta}{2}\right)$  identity, we begin with the PRI:

$$\sin^2 u = \frac{1 - \cos(2u)}{2}$$

$$\text{Let } u = \frac{\theta}{2}, \text{ or } \theta = 2u.$$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{2}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos\theta}{2}} \quad (\text{by the Square Root Method})$$

Again, the choice among the  $\pm$  signs depends on the Quadrant that  $\frac{\theta}{2}$  lies in.

The story is similar for the  $\cos\left(\frac{\theta}{2}\right)$  and the  $\tan\left(\frac{\theta}{2}\right)$  identities.

What about the last two formulas for  $\tan\left(\frac{\theta}{2}\right)$ ? The key trick is multiplication by trig conjugates. For example:

$$\begin{aligned} \tan\left(\frac{\theta}{2}\right) &= \pm \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} \\ &= \pm \sqrt{\frac{(1 - \cos\theta)(1 - \cos\theta)}{(1 + \cos\theta)(1 - \cos\theta)}} \\ &= \pm \sqrt{\frac{(1 - \cos\theta)^2}{1 - \cos^2\theta}} \\ &= \pm \sqrt{\frac{(1 - \cos\theta)^2}{\sin^2\theta}} \\ &= \pm \sqrt{\left(\frac{1 - \cos\theta}{\sin\theta}\right)^2} \\ &= \pm \left| \frac{1 - \cos\theta}{\sin\theta} \right| \quad \left( \text{because } \sqrt{blah^2} = |blah| \right) \end{aligned}$$

Now,  $1 - \cos \theta \geq 0$  for all real  $\theta$ , and  $\tan\left(\frac{\theta}{2}\right)$  has the same sign as  $\sin \theta$  (can you see why?), so ...

$$= \frac{1 - \cos \theta}{\sin \theta}$$

To get the third formula, use the numerator's (instead of the denominator's) trig conjugate,  $1 + \cos \theta$ , when multiplying into the numerator and the denominator of the radicand in the first few steps.

### **PRODUCT-TO-SUM IDENTITIES** (Given as necessary on exams)

These can be verified from right-to-left using the Sum and Difference Identities.

#### **The Identities:**

$$\sin u \sin v = \frac{1}{2} [\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2} [\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2} [\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2} [\sin(u + v) - \sin(u - v)]$$

### **SUM-TO-PRODUCT IDENTITIES** (Given as necessary on exams)

These can be verified from right-to-left using the Product-To-Sum Identities.

#### **The Identities:**

$$\sin x + \sin y = 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$

$$\sin x - \sin y = 2 \cos\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)$$