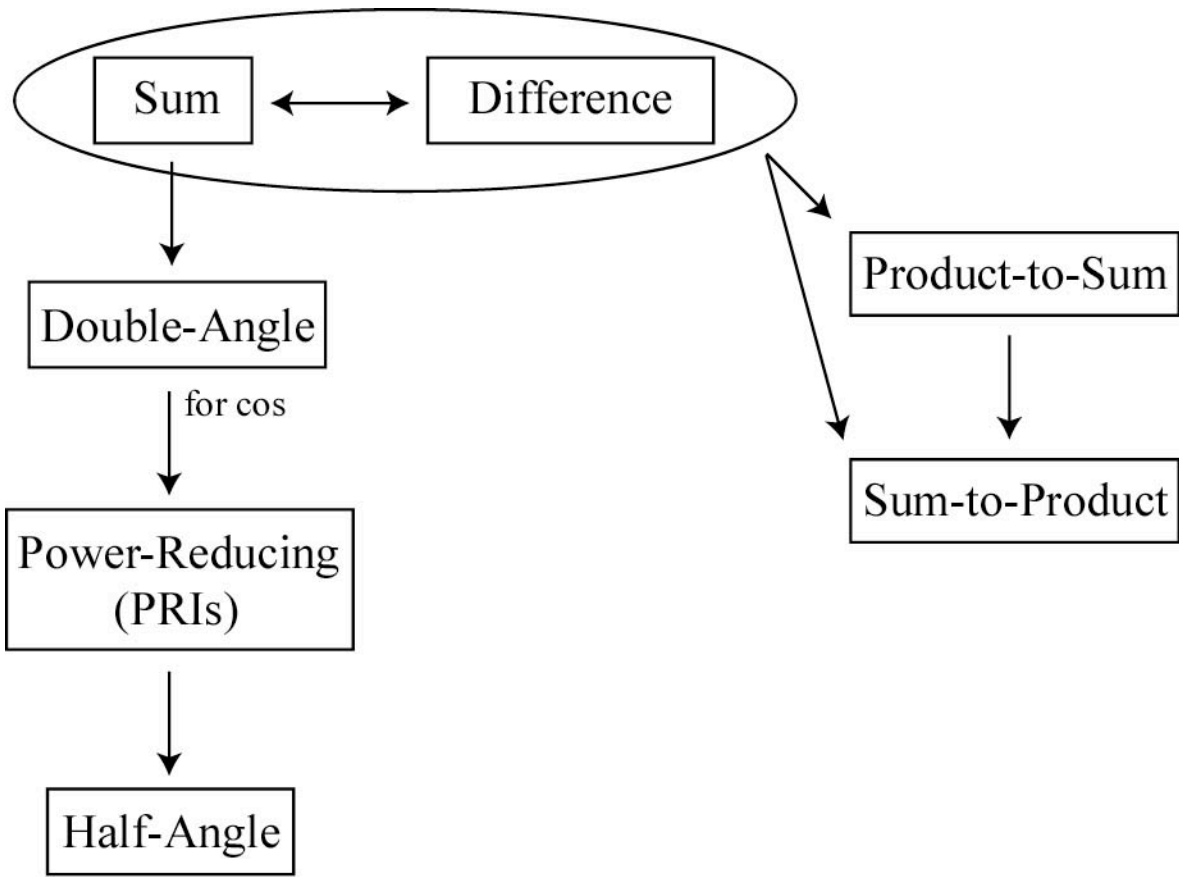
SECTIONS 5.4 and 5.5: MORE TRIG IDENTITIES

PART A: A GUIDE TO THE HANDOUT

See the Handout on my website.

The identities (IDs) may be derived according to this flowchart:



In Calculus: The Double-Angle and Power-Reducing IDs are most commonly used among these, though we will discuss a critical application of the Sum IDs in Part C.

Some proofs are on pp.403-5. See p.381 for notes on Hipparchus, the "inventor" of trig, and the father of the Sum and Difference IDs.

PART B: EXAMPLES

Example: Finding Trig Values

Find the exact value of $\sin 15^\circ$.

Note: Larson uses radians to solve this in Example 2 on p.381, but degrees are usually easier to deal with when applying these identities, since we don't have to worry about common denominators.

Solution (Method 1: Difference ID)

We know trig values for 45° and 30°, so a Difference ID should work.

$$\sin 15^\circ = \sin\left(45^\circ - 30^\circ\right)$$

Use: $\sin\left(u - v\right) = \sin u \cos v - \cos u \sin v$

$$\begin{aligned} &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

Warning: $\sqrt{6} - \sqrt{2} \neq \sqrt{4}$. We do **not** have sum and difference rules for radicals the same way we have product and quotient rules for them.

Solution (Method 2: Half-Angle ID)

We know trig values for 30°, so a Half-Angle ID should work.

$$\sin 15^\circ = \sin\left(\frac{30^\circ}{2}\right)$$

Use: $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$

$$= \pm\sqrt{\frac{1-\cos 30^\circ}{2}}$$

$$= \pm\sqrt{\frac{\left(1-\frac{\sqrt{3}}{2}\right)}{2}\cdot\frac{2}{2}}$$

$$= \pm\sqrt{\frac{2-\sqrt{3}}{4}}$$

$$= \pm\frac{\sqrt{2-\sqrt{3}}}{2}$$

We know $\sin 15^\circ > 0$, since 15° is an acute Quadrant I angle. We take the "+" sign.

$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

In fact, $\frac{\sqrt{2-\sqrt{3}}}{2}$ is equivalent to $\frac{\sqrt{6}-\sqrt{2}}{4}$, our result from Method 1. They are both positive in value, and you can see (after some work) that their squares are equal.

Example: Simplifying and/or Evaluating

Find the exact value of: $\dfrac{\tan 25^\circ + \tan 20^\circ}{1 - \tan 25^\circ \tan 20^\circ}$

Solution

We do not know the exact tan values for 25° or 20°, but observe that the expression follows the template for the Sum Formula for tan:

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

We will use this ID “in reverse” (i.e., from right-to-left):

$$\frac{\tan u + \tan v}{1 - \tan u \tan v} = \tan(u+v)$$

$$\begin{aligned} \frac{\tan 25^\circ + \tan 20^\circ}{1 - \tan 25^\circ \tan 20^\circ} &= \tan(25^\circ + 20^\circ) \\ &= \tan 45^\circ \\ &= 1 \end{aligned}$$

Example: Simplifying Trig Expressions

Simplify: $\dfrac{1}{\sin(3\theta)\cos(3\theta)}$

Solution

We will take the Double-Angle ID: $\sin(2u) = 2\sin u \cos u$ and use it "in reverse": $2\sin u \cos u = \sin(2u)$.

Let $u = 3\theta$. Observe:

$$\begin{aligned} 2\sin(3\theta)\cos(3\theta) &= \sin[2(3\theta)] \\ 2\sin(3\theta)\cos(3\theta) &= \sin(6\theta) \\ \sin(3\theta)\cos(3\theta) &= \frac{1}{2}\sin(6\theta) \end{aligned}$$

Note: We also get this result from the Product-to-Sum Identities, but they are harder to remember!

Therefore,

$$\begin{aligned} \frac{1}{\sin(3\theta)\cos(3\theta)} &= \frac{1}{\frac{1}{2}\sin(6\theta)} \\ &= 2\csc(6\theta) \end{aligned}$$

Examples: Verifying Trig IDs

Examples 5 and 6 on p.382 of Larson show how these IDs can be used to verify Cofunction IDs and Reduction IDs.

Example: These IDs can be used to verify something like: $\sin(\theta + \pi) = -\sin\theta$. Can you see why this is true using the Unit Circle?

Examples: Solving Trig Equations

See Example 8 on p.383 of Larson.

Example

Solve: $\sin x - \cos(2x) = 0$

Solution

We will use the Double-Angle ID for $\cos(2x)$.

$$\begin{aligned}\sin x - \cos(2x) &= 0\\ \sin x - \left(\cos^2 x - \sin^2 x\right) &= 0\\ \sin x - \cos^2 x + \sin^2 x &= 0\end{aligned}$$

Warning: Remember to use grouping symbols if you are subtracting a substitution result consisting of more than one term.

Use the basic Pythagorean Identity to express $\cos^2 x$ in terms of a power of $\sin x$.

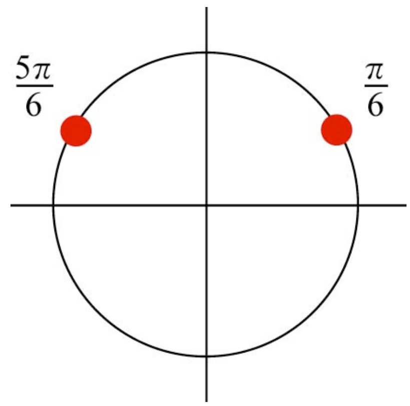
$$\begin{aligned}\sin x - \left(1 - \sin^2 x\right) + \sin^2 x &= 0\\ \sin x - 1 + \sin^2 x + \sin^2 x &= 0\\ 2\sin^2 x + \sin x - 1 &= 0\end{aligned}$$

You can use the substitution $u = \sin x$, or you can factor directly.

$$(2\sin x - 1)(\sin x + 1) = 0$$

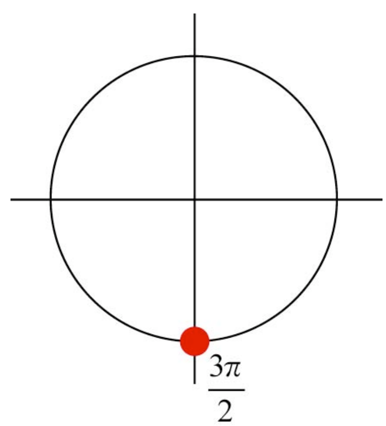
<u>First factor</u>:

$$2\sin x - 1 = 0$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6} + 2\pi n, \text{ or } x = \frac{5\pi}{6} + 2\pi n \quad (n \text{ integer})$$



<u>Second factor</u>:

$$\sin x + 1 = 0$$
$$\sin x = -1$$
$$x = \frac{3\pi}{2} + 2\pi n \quad (n \text{ integer})$$

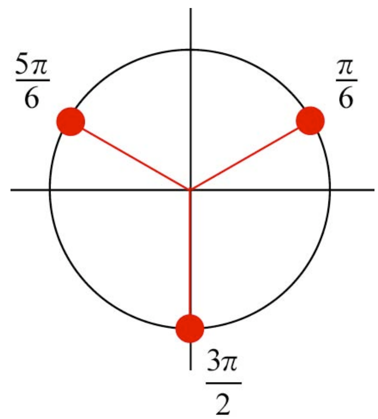


<u>Solution set</u>:

$$\left\{ x \,\middle|\, x = \frac{\pi}{6} + 2\pi n, \; x = \frac{5\pi}{6} + 2\pi n, \text{ or } x = \frac{3\pi}{2} + 2\pi n \quad (n \text{ integer}) \right\}$$

A More Efficient Form!

Look at the red points (corresponding to solutions) we've collected on the Unit Circle:



The solutions exhibit a "period" of $\frac{2\pi}{3}$, corresponding to "third-revolutions" about the Unit Circle.

Here is a much more efficient form for the solution set:

$$\left\{ x \,\middle|\, x = \frac{\pi}{6} + \frac{2\pi}{3}n \quad (n \text{ integer}) \right\}$$

Examples: Using Right Triangles

Example

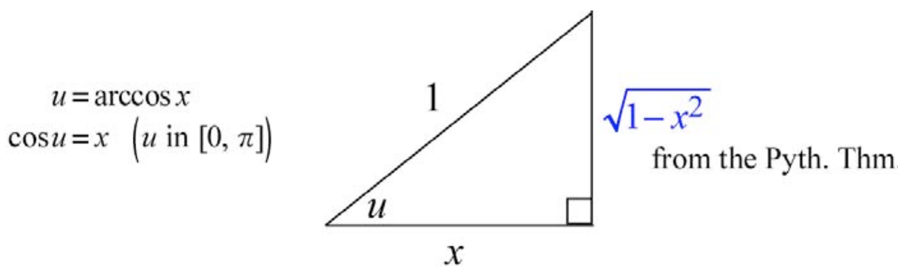
Express $\sin(2\arccos x)$ as an equivalent algebraic expression in x.
Assume x is in $[-1,1]$, the domain of the arccos function.

Solution

We use the Double-Angle ID: $\sin(2u)=2\sin u\cos u$, where $u=\arccos x$.

$$\sin(2\arccos x)=2\sin(\arccos x)\cos(\arccos x)$$

Since x is assumed to be in $[-1,1]$, we know that $\cos(\arccos x)=x$.
We will use a right triangle model to reexpress $\sin(\arccos x)$.



$$\sin(\arccos x)=\frac{\sqrt{1-x^2}}{1}=\sqrt{1-x^2}$$

We then have ... $\sin(2\arccos x)=2\left(\sqrt{1-x^2}\right)(x)$

$$=2x\sqrt{1-x^2}$$

See Example 4 on p.381 of Larson. When rewriting $\cos(\arctan 1+\arccos x)$, we let $u=\arctan 1$ and $v=\arccos x$, and we can apply the Sum Identity for $\cos(u+v)$.

It may help to recognize that $\arctan 1=\dfrac{\pi}{4}$.

Examples: Using Power-Reducing IDs (PRIs)

See Example 5 on p.389. In Calculus: You will need to do this when you do advanced techniques of integration in Calculus II: Math 151 at Mesa. In the next Example, we will explain one of the more confusing steps in the solution:

Example

Express $\cos^2(2x)$ in terms of first powers of cosines.

Solution

We use the PRI: $\cos^2 u = \dfrac{1+\cos(2u)}{2}$, where $u = 2x$.

$$\cos^2(2x) = \frac{1+\cos\left[2(2x)\right]}{2} = \frac{1+\cos(4x)}{2}$$

Example: Product-to-Sum ID

Example

Apply a Product-to-Sum ID to reexpress $\sin(6\theta)\sin(4\theta)$ as an equivalent expression.

Solution

The relevant ID is: $\sin u \sin v = \frac{1}{2}\left[\cos(u-v)-\cos(u+v)\right]$

$$\sin(6\theta)\sin(4\theta) = \frac{1}{2}\left[\cos(6\theta-4\theta)-\cos(6\theta+4\theta)\right] = \frac{1}{2}\left[\cos(2\theta)-\cos(10\theta)\right]$$

Example: Sum-to-Product ID

Example

Apply a Sum-to-Product ID to reexpress $\sin(6\theta)+\sin(4\theta)$ as an equivalent expression.

Warning: This is **not** equivalent to $\sin(10\theta)$.

Solution

The relevant ID is: $\sin x+\sin y=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$$\begin{aligned}\sin(6\theta)+\sin(4\theta)&=2\sin\left(\frac{6\theta+4\theta}{2}\right)\cos\left(\frac{6\theta-4\theta}{2}\right)\\&=2\sin\left(\frac{10\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)\\&=2\sin(5\theta)\cos\theta\end{aligned}$$

Example: Extending IDs

Example 4 on p.389 in Larson shows how a Triple-Angle ID can be derived from the Double-Angle IDs.

PART C: APPLICATIONS IN CALCULUS

Review difference quotients and derivatives ("slope functions") in Notes 1.57 and 1.58.

The Sum IDs help us show that:

If $f(x) = \sin x$, then the derivative $f'(x) = \cos x$.
If $f(x) = \cos x$, then the derivative $f'(x) = -\sin x$.

Let's consider $f(x) = \sin x$. We will use a limit definition for the derivative:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

We will use a Sum ID to expand $\sin(x+h)$.

Example 7 on p.383 works this out in a slightly different way.

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h - \sin x) + \cos x \sin h}{h} \qquad \text{(Group terms with } \sin x.\text{)}$$

$$= \lim_{h \to 0} \frac{(\sin x)(\cos h - 1) + \cos x \sin h}{h} \qquad \text{(Factor } \sin x \text{ out of the group.)}$$

$$= \lim_{h \to 0} \left[(\sin x) \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\to 0} + (\cos x) \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right]$$

As $h \to 0$, $\frac{\cos h - 1}{h} \to 0$, or, equivalently, $\frac{1 - \cos h}{h} \to 0$, if you use the book's result.

Also, $\frac{\sin h}{h} \to 1$.

$$= \cos x$$