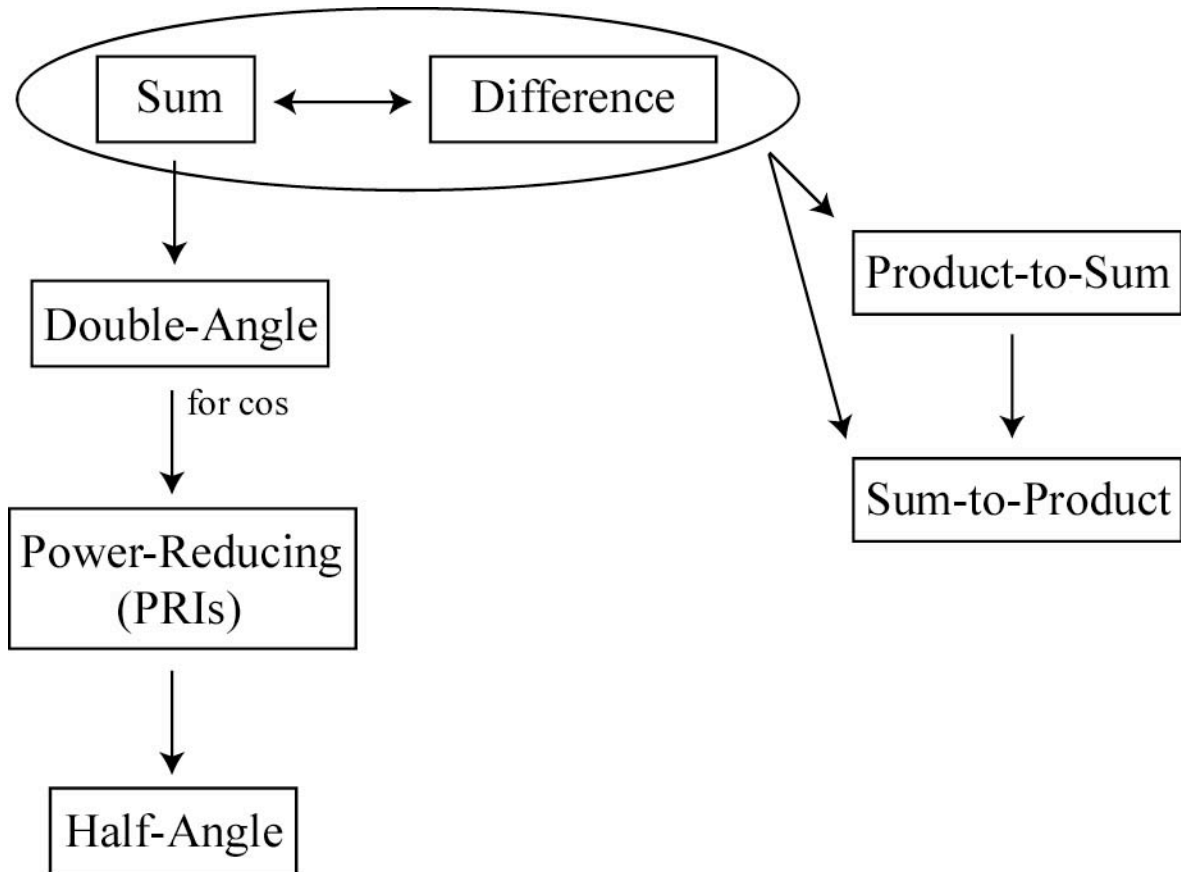


## SECTIONS 5.4 and 5.5: MORE TRIG IDENTITIES

### PART A: A GUIDE TO THE HANDOUT

[See the Handout on my website.](#)

The identities (IDs) may be derived according to this flowchart:



In Calculus: The Double-Angle and Power-Reducing IDs are most commonly used among these, though we will discuss a critical application of the Sum IDs in [Part C](#).

Some proofs are on [pp.403-5](#). See [p.381](#) for notes on Hipparchus, the “inventor” of trig, and the father of the Sum and Difference IDs.

**PART B: EXAMPLES**Example: Finding Trig Values

Find the exact value of  $\sin 15^\circ$ .

Note: Larson uses radians to solve this in [Example 2 on p.381](#), but degrees are usually easier to deal with when applying these identities, since we don't have to worry about common denominators.

Solution (Method 1: Difference ID)

We know trig values for  $45^\circ$  and  $30^\circ$ , so a Difference ID should work.

$$\sin 15^\circ = \sin(45^\circ - 30^\circ)$$

$$\text{Use: } \sin(u - v) = \sin u \cos v - \cos u \sin v$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Warning:  $\sqrt{6} - \sqrt{2} \neq \sqrt{4}$ . We do **not** have sum and difference rules for radicals the same way we have product and quotient rules for them.

Solution (Method 2: Half-Angle ID)

We know trig values for  $30^\circ$ , so a Half-Angle ID should work.

$$\sin 15^\circ = \sin\left(\frac{30^\circ}{2}\right)$$

$$\text{Use: } \sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$\begin{aligned} &= \pm \sqrt{\frac{1 - \cos 30^\circ}{2}} \\ &= \pm \sqrt{\frac{\left(1 - \frac{\sqrt{3}}{2}\right)}{2} \cdot \frac{2}{2}} \\ &= \pm \sqrt{\frac{2 - \sqrt{3}}{4}} \\ &= \pm \frac{\sqrt{2 - \sqrt{3}}}{2} \end{aligned}$$

We know  $\sin 15^\circ > 0$ , since  $15^\circ$  is an acute Quadrant I angle. We take the “+” sign.

$$= \frac{\sqrt{2 - \sqrt{3}}}{2}$$

In fact,  $\frac{\sqrt{2 - \sqrt{3}}}{2}$  is equivalent to  $\frac{\sqrt{6} - \sqrt{2}}{4}$ , our result from Method 1.

They are both positive in value, and you can see (after some work) that their squares are equal.

Example: Simplifying and/or Evaluating

Find the exact value of:  $\frac{\tan 25^\circ + \tan 20^\circ}{1 - \tan 25^\circ \tan 20^\circ}$

Solution

We do not know the exact tan values for  $25^\circ$  or  $20^\circ$ , but observe that the expression follows the template for the Sum Formula for tan:

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

We will use this ID “in reverse” (i.e., from right-to-left):

$$\frac{\tan u + \tan v}{1 - \tan u \tan v} = \tan(u + v)$$

$$\begin{aligned} \frac{\tan 25^\circ + \tan 20^\circ}{1 - \tan 25^\circ \tan 20^\circ} &= \tan(25^\circ + 20^\circ) \\ &= \tan 45^\circ \\ &= 1 \end{aligned}$$

Example: Simplifying Trig Expressions

Simplify:  $\frac{1}{\sin(3\theta)\cos(3\theta)}$

Solution

We will take the Double-Angle ID:  $\sin(2u) = 2 \sin u \cos u$  and use it “in reverse”:  $2 \sin u \cos u = \sin(2u)$ .

Let  $u = 3\theta$ . Observe:

$$\begin{aligned} 2 \sin(3\theta)\cos(3\theta) &= \sin[2(3\theta)] \\ 2 \sin(3\theta)\cos(3\theta) &= \sin(6\theta) \\ \sin(3\theta)\cos(3\theta) &= \frac{1}{2} \sin(6\theta) \end{aligned}$$

Note: We also get this result from the Product-to-Sum Identities, but they are harder to remember!

Therefore,

$$\begin{aligned} \frac{1}{\sin(3\theta)\cos(3\theta)} &= \frac{1}{\frac{1}{2} \sin(6\theta)} \\ &= 2 \csc(6\theta) \end{aligned}$$

Examples: Verifying Trig IDs

[Examples 5 and 6 on p.382 of Larson](#) show how these IDs can be used to verify Cofunction IDs and Reduction IDs.

Example: These IDs can be used to verify something like:  $\sin(\theta + \pi) = -\sin \theta$ .

Can you see why this is true using the Unit Circle?

Examples: Solving Trig Equations[See Example 8 on p.383 of Larson.](#)Example

Solve:  $\sin x - \cos(2x) = 0$

SolutionWe will use the Double-Angle ID for  $\cos(2x)$ .

$$\begin{aligned}\sin x - \cos(2x) &= 0 \\ \sin x - (\cos^2 x - \sin^2 x) &= 0 \\ \sin x - \cos^2 x + \sin^2 x &= 0\end{aligned}$$

**Warning: Remember to use grouping symbols if you are subtracting a substitution result consisting of more than one term.**Use the basic Pythagorean Identity to express  $\cos^2 x$  in terms of a power of  $\sin x$ .

$$\begin{aligned}\sin x - (1 - \sin^2 x) + \sin^2 x &= 0 \\ \sin x - 1 + \sin^2 x + \sin^2 x &= 0 \\ 2\sin^2 x + \sin x - 1 &= 0\end{aligned}$$

You can use the substitution  $u = \sin x$ , or you can factor directly.

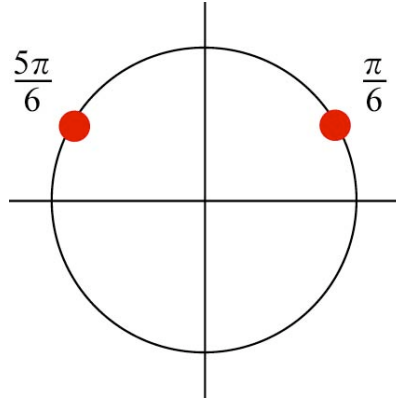
$$(2\sin x - 1)(\sin x + 1) = 0$$

First factor:

$$2 \sin x - 1 = 0$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} + 2\pi n, \text{ or } x = \frac{5\pi}{6} + 2\pi n \quad (n \text{ integer})$$

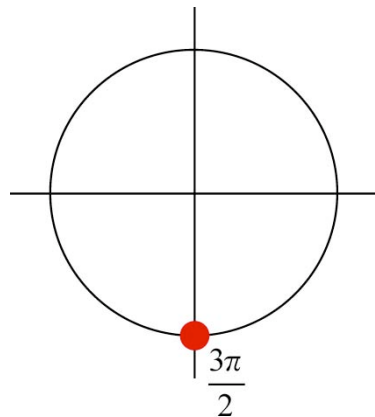


Second factor:

$$\sin x + 1 = 0$$

$$\sin x = -1$$

$$x = \frac{3\pi}{2} + 2\pi n \quad (n \text{ integer})$$

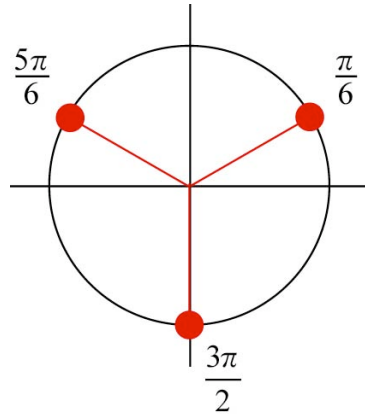


Solution set:

$$\left\{ x \mid x = \frac{\pi}{6} + 2\pi n, x = \frac{5\pi}{6} + 2\pi n, \text{ or } x = \frac{3\pi}{2} + 2\pi n \quad (n \text{ integer}) \right\}$$

A More Efficient Form!

Look at the red points (corresponding to solutions) we've collected on the Unit Circle:



The solutions exhibit a “period” of  $\frac{2\pi}{3}$ , corresponding to “third-revolutions” about the Unit Circle.

Here is a much more efficient form for the solution set:

$$\left\{ x \mid x = \frac{\pi}{6} + \frac{2\pi}{3} n \quad (n \text{ integer}) \right\}$$

Examples: Using Right TrianglesExample

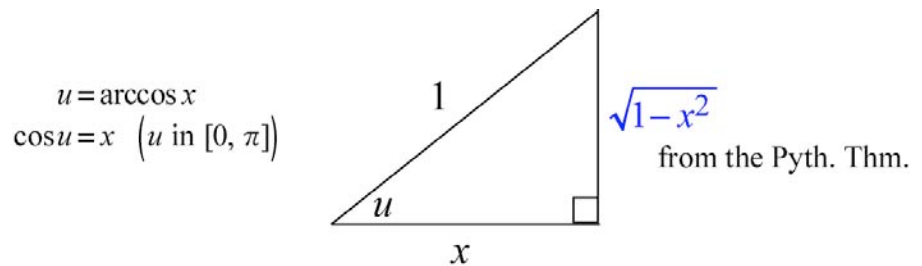
Express  $\sin(2 \arccos x)$  as an equivalent algebraic expression in  $x$ .  
Assume  $x$  is in  $[-1, 1]$ , the domain of the arccos function.

Solution

We use the Double-Angle ID:  $\sin(2u) = 2 \sin u \cos u$ , where  $u = \arccos x$ .

$$\sin(2 \arccos x) = 2 \sin(\arccos x) \cos(\arccos x)$$

Since  $x$  is assumed to be in  $[-1, 1]$ , we know that  $\cos(\arccos x) = x$ .  
We will use a right triangle model to reexpress  $\sin(\arccos x)$ .



$$\sin(\arccos x) = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

$$\begin{aligned} \text{We then have ... } \sin(2 \arccos x) &= 2(\sqrt{1-x^2})(x) \\ &= 2x\sqrt{1-x^2} \end{aligned}$$

[See Example 4 on p.381 of Larson.](#) When rewriting  $\cos(\arctan 1 + \arccos x)$ , we let  $u = \arctan 1$  and  $v = \arccos x$ , and we can apply the Sum Identity for  $\cos(u + v)$ .

It may help to recognize that  $\arctan 1 = \frac{\pi}{4}$ .

Examples: Using Power-Reducing IDs (PRIs)

[See Example 5 on p.389. In Calculus:](#) You will need to do this when you do advanced techniques of integration in [Calculus II: Math 151 at Mesa](#). In the next Example, we will explain one of the more confusing steps in the solution:

Example

Express  $\cos^2(2x)$  in terms of first powers of cosines.

Solution

We use the PRI:  $\cos^2 u = \frac{1 + \cos(2u)}{2}$ , where  $u = 2x$ .

$$\begin{aligned}\cos^2(2x) &= \frac{1 + \cos[2(2x)]}{2} \\ &= \frac{1 + \cos(4x)}{2}\end{aligned}$$

Example: Product-to-Sum IDExample

Apply a Product-to-Sum ID to reexpress  $\sin(6\theta)\sin(4\theta)$  as an equivalent expression.

Solution

The relevant ID is:  $\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$

$$\begin{aligned}\sin(6\theta)\sin(4\theta) &= \frac{1}{2}[\cos(6\theta - 4\theta) - \cos(6\theta + 4\theta)] \\ &= \frac{1}{2}[\cos(2\theta) - \cos(10\theta)]\end{aligned}$$

Example: Sum-to-Product IDExample

Apply a Sum-to-Product ID to reexpress  $\sin(6\theta) + \sin(4\theta)$  as an equivalent expression.

Warning: This is **not** equivalent to  $\sin(10\theta)$ .

Solution

The relevant ID is:  $\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$

$$\begin{aligned}\sin(6\theta) + \sin(4\theta) &= 2 \sin\left(\frac{6\theta + 4\theta}{2}\right) \cos\left(\frac{6\theta - 4\theta}{2}\right) \\ &= 2 \sin\left(\frac{10\theta}{2}\right) \cos\left(\frac{2\theta}{2}\right) \\ &= 2 \sin(5\theta) \cos\theta\end{aligned}$$

Example: Extending IDs

[Example 4 on p.389 in Larson](#) shows how a Triple-Angle ID can be derived from the Double-Angle IDs.

**PART C: APPLICATIONS IN CALCULUS**

Review difference quotients and derivatives (“slope functions”) in [Notes 1.57 and 1.58](#).

The Sum IDs help us show that:

$$\begin{aligned} \text{If } f(x) = \sin x, \text{ then the derivative } f'(x) &= \cos x. \\ \text{If } f(x) = \cos x, \text{ then the derivative } f'(x) &= -\sin x. \end{aligned}$$

Let’s consider  $f(x) = \sin x$ . We will use a limit definition for the derivative:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$$

We will use a Sum ID to expand  $\sin(x+h)$ .

[Example 7 on p.383](#) works this out in a slightly different way.

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sin x \cos h - \sin x) + \cos x \sin h}{h} && \text{(Group terms with } \sin x \text{.)} \\ &= \lim_{h \rightarrow 0} \frac{(\sin x)(\cos h - 1) + \cos x \sin h}{h} && \text{(Factor } \sin x \text{ out of the group.)} \\ &= \lim_{h \rightarrow 0} \left[ (\sin x) \underbrace{\left( \frac{\cos h - 1}{h} \right)}_{\rightarrow 0} + (\cos x) \underbrace{\left( \frac{\sin h}{h} \right)}_{\rightarrow 1} \right] \end{aligned}$$

As  $h \rightarrow 0$ ,  $\frac{\cos h - 1}{h} \rightarrow 0$ , or, equivalently,  $\frac{1 - \cos h}{h} \rightarrow 0$ , if you use the book’s result.

$$\text{Also, } \frac{\sin h}{h} \rightarrow 1.$$

$$= \cos x$$