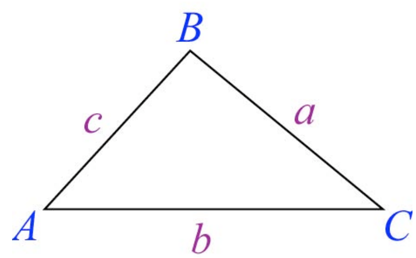
# CHAPTER 6: ADDITIONAL TOPICS IN TRIG

## SECTION 6.1: THE LAW OF SINES

### PART A: THE SETUP AND THE LAW

The Law of Sines and the Law of Cosines will allow us to analyze and solve oblique (i.e., non-right) triangles, as well as the right triangles we have been used to dealing with.

Here is an example of a conventional setup for a triangle:




There are 6 parts: 3 angles and 3 sides.

Observe that Side $a$ "faces" Angle $A$, $b$ faces $B$, and $c$ faces $C$.
(In a right triangle, $C$ was typically the right angle.)

When we refer to $a$, we may be referring to the line segment $\overline{BC}$ or its length.

The Law of Sines

For such a triangle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Equivalently:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

## PART B: WHAT MUST BE TRUE OF ALL TRIANGLES?

We assume that *A*, *B*, and *C* are angles whose degree measures are strictly between $0^\circ$ and $180^\circ$.

**The $180^\circ$ Rule**

The sum of the interior angles of a triangle must be $180^\circ$. That is, $A + B + C = 180^\circ$.

**The Triangle Inequality**

The sum of any two sides (i.e., side lengths) of a triangle must exceed the third. That is, $a + b > c$, $b + c > a$, and $a + c > b$.

Think: Detours. Any detour in the plane from point *A* to point *B*, for example, must be longer than the straight route from *A* to *B*.

Example: There can be no triangle with side lengths 3 cm, 4 cm, and 10 cm, because $3 + 4 \not> 10$. If you had three "pick-up" sticks with those lengths, you could not form a triangle with them if you were only allowed to connect them at their endpoints.

**The "Eating" Rule**

For a given triangle, larger angles face (or "eat") longer sides.

You can use this to check to see if your answers are sensible.

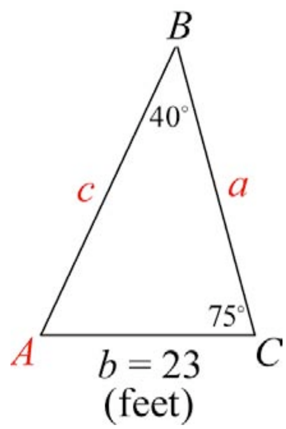
**PART C: EXAMPLE**

Example

Given: $B = 40^\circ$, $C = 75^\circ$, $b = 23$ ft.
Solve the triangle. In your final answers, round off lengths to the nearest foot.

Solution

Sketch a model triangle.
(Information yet to be determined is in red.)




Find Angle $A$:

Use the $180^\circ$ Rule.

$$A = 180^\circ - 40^\circ - 75^\circ$$
$$\boldsymbol{A = 65^\circ}$$

Use the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 65^\circ} = \frac{23}{\sin 40^\circ} = \frac{c}{\sin 75^\circ}$$

Observe that the middle ratio is "known," so we should use it when we solve for both $a$ and $c$.

Find $a$:

Solve $\dfrac{a}{\sin 65^\circ} = \dfrac{23}{\sin 40^\circ}$ for $a$.

$$a = \frac{23 \sin 65^\circ}{\sin 40^\circ}$$

$$\boldsymbol{a \approx 32 \text{ feet}}$$

Warning: Make sure your calculator is in DEGREE mode.

Warning: Avoid approximations for trig values in intermediate steps. Excessive rounding can render your final approximations inaccurate. Memory buttons may help. If feasible, you should keep exact expressions such as $\sin 65^\circ$ throughout your solution until the end.

Warning: Don't forget units where they are appropriate!

Find $c$:

Solve $\dfrac{23}{\sin 40^\circ} = \dfrac{c}{\sin 75^\circ}$ for $c$.

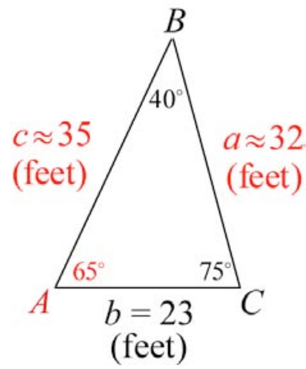
$$\frac{23 \sin 75^\circ}{\sin 40^\circ} = c$$

$$\boldsymbol{c \approx 35 \text{ feet}}$$

Warning: Although you could, in principle, use the $\dfrac{a}{\sin 65^\circ}$ ratio instead of the $\dfrac{23}{\sin 40^\circ}$ ratio, it is ill-advised to use your rough approximation for $a$ as a foundation for your new calculations.

Use the "Eating" Rule to Check:

Make sure that larger angles "eat" longer sides.




## PART D: CASES

If you are given 3 parts of a triangle (including one side), you can solve the triangle by finding the other 3 parts or by discovering that there is no triangle that supports the given configuration (in which case there is "no solution"). Two triangles are considered to be the same if they have the same values for $A$, $B$, $C$, $a$, $b$, and $c$.

• If you are only given the 3 angles (the AAA case), then you have an entire family of similar triangles of varying sizes that have those angles.

• On the other hand, it is possible that **no** triangle can support the given configuration. For example, maybe the Triangle Inequality is being violated.

• In the "Ambiguous SSA" case, which is discussed in Part E, two different triangles may work.

The Law of Sines is applied in cases where you know two angles and one side.

For example, it is applied in:

• The AAS case (such as in the previous Example, in which we are given Angle $B$, Angle $C$, and Side $b$, a "nonincluded" side), and

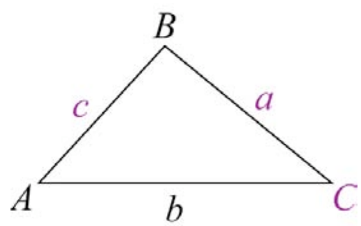
• The ASA case (in which, for example, we are given Angle $A$, Side $b$, and Angle $C$; here, Side $b$ is "included" between the two given angles).

Can you see how these cases can yield either no triangle or exactly one?

The Law of Sines is also applied in the "Ambiguous SSA" case, described next.

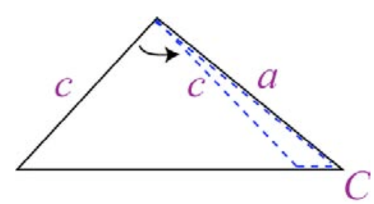
**PART E: THE AMBIGUOUS SSA CASE**

The SSA case is when you are given two sides and a nonincluded angle.
For example, you could be given Side $c$, Side $a$, and Angle $C$ (in purple below).




The SSA case is called the "ambiguous case," because two triangles (that is, two triangles that are not congruent) may arise from the given information. Bear in mind that the possibility of "no triangles" potentially plagues all cases.

In the figure above, imagine the side labeled $c$ being rotated about Point $B$.
We can obtain a second triangle (in blue dashed lines below) for which the given information still holds!




How is this issue reflected in the Law of Sines?

Look at the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In order to solve the triangle, we find Angle $A$ using the Law of Sines. We could then use the $180^\circ$ Rule to find the remaining angle, Angle $B$.

The problem is that we must first find $\sin A$, and an acute angle and its supplementary obtuse angle may share the same sin value. (Look at the Unit Circle!) These two possibilities may yield two different triangles, provided that the $180^\circ$ Rule does not fall apart (the obtuse angle may "eat up" too many degrees). Also, it's "game over" if $\sin A$ was not in $(0,1]$ to begin with.

## PART F: THE AREA OF A TRIANGLE

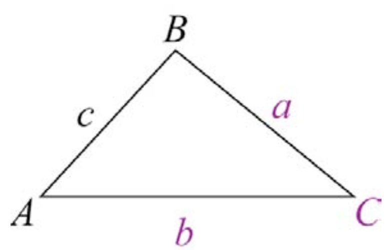
In the SAS Case (described below), you can quickly compute the area of a triangle.

Area of a Triangle (SAS Case)

Let $a$ and $b$ be two sides (i.e., two side lengths) of a triangle, and let $C$ be the included angle between them. Then, the area of the triangle is given by:

$$\text{Area} = \frac{1}{2}ab\sin C$$

Think: Half the product of two sides and the sine of the included angle between them (represented by a different letter).




Warning: Avoid writing $A$ for Area, since we often use $A$ to name a vertex on the triangle.

What happens if $C$ is a right angle?

Variations

The following also hold:

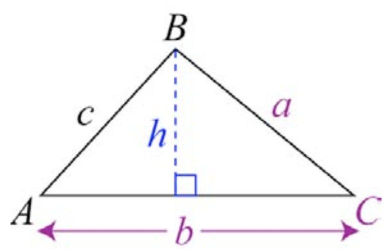
$$\text{Area} = \frac{1}{2}bc\sin A$$

$$\text{Area} = \frac{1}{2}ac\sin B$$

If you know one of the area formulas above, you can figure out the other two. If you are given a word problem or an unlabeled triangle, you could assign labels in a manner best suited for the formula you are most familiar with.

Proof

"Without loss of generality," let's say we are given $a$, $b$, and $C$ in the figure below. The $h$ represents the height of the triangle, provided that $b$ is taken as the base.




Since $\sin C = \dfrac{h}{a}$, the height $h$ of the triangle is given by: $h = a \sin C$.
The area is then given by:

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}bh = \frac{1}{2}b(a\sin C) = \frac{1}{2}ab\sin C$$

This proof is similar to the proof for the Law of Sines on p.468 of Larson, in which expressions for the height of a triangle involving the sines of the three angles are equated.