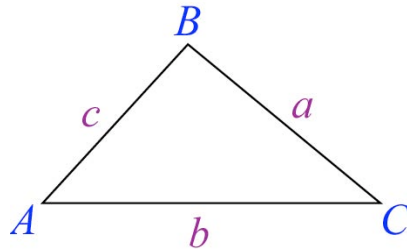


## SECTION 6.2: THE LAW OF COSINES

### PART A: THE SETUP AND THE LAW

Remember our example of a conventional setup for a triangle:



Observe that Side  $a$  “faces” Angle  $A$ ,  $b$  faces  $B$ , and  $c$  faces  $C$ .

#### The Law of Cosines

For such a triangle:

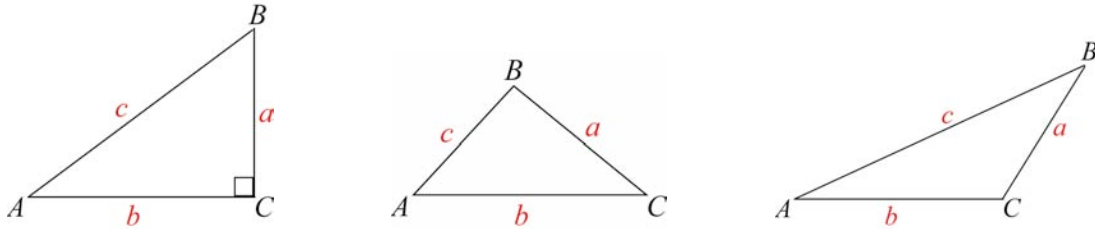
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Think / For Memorization Purposes: This looks like the Pythagorean Theorem, except that there is a third term. “The square of one side equals the sum of the squares of the other two sides, **minus** twice their product times the cosine of the angle included between them.” Notice that the formula is symmetric in  $a$  and  $b$ ; we have  $2ab$  in the formula as opposed to  $2bc$  or  $2ac$ . Angle  $C$  is the one we take the cosine of, because it is the “special” angle that faces the side indicated on the left.

The (very involved) proof is on [p.469 of Larson](#). There is a nicer “Proof Without Words” available on my website.

**PART B: A THOUGHT EXPERIMENT**

How does the formula gibe with our geometric intuition?



Let's say we fix lengths  $a$  and  $b$ , but we allow the other parts to vary. Imagine rotating the side labeled  $a$  about the point  $C$  so that Angle  $C$  changes.

If  $C$  is a right angle (left figure above), then we obtain the Pythagorean Theorem as a special case:

$$c^2 = a^2 + b^2 - \underbrace{2ab \cos\left(\frac{\pi}{2}\right)}_{=0}$$

$$c^2 = a^2 + b^2$$

If  $C$  is acute (middle figure above), then:

$$c^2 = a^2 + b^2 - \underbrace{2ab \cos C}_{<0}$$

$$c^2 < a^2 + b^2$$

This reflects the fact that  $c$  in this case is smaller than  $c$  in the right angle case.

If  $C$  is obtuse (right figure above), then:

$$c^2 = a^2 + b^2 - \underbrace{2ab \cos C}_{>0}$$

$$c^2 > a^2 + b^2$$

This reflects the fact that  $c$  in this case is larger than  $c$  in the right angle case.

**PART C: VARIATIONS OF THE LAW**

The form given in [Part A](#) is the only one you need to memorize, but you should be aware of variations.

There is nothing “special” about side  $c$  and Angle  $C$ . “Role-switching” yields analogous formulas for the other side-angle pairs.

Variations of the Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

Without loss of generality, the proof of one variation yields the others, as well.

The previous formulas may also be solved for the cos expressions.

More Variations of the Law of Cosines

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

In radians,  $A$ ,  $B$ , and  $C$  must be in the interval  $(0, \pi)$ , which is a subset of the range of the  $\cos^{-1}$  function,  $[0, \pi]$ . Therefore, we can conclude that:

$$C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right), \text{ and so forth. However, books tend not to give these as}$$

variations of the Law of Cosines, because, if we are working in degrees, they may be deemed inappropriate. (Inverse trig values are not supposed to be directly given in degrees.) As a practical matter, though, it helps to be aware of them.

**PART D: WHAT MUST BE TRUE OF ALL TRIANGLES?**

We assume that  $A$ ,  $B$ , and  $C$  are angles whose degree measures are strictly between  $0^\circ$  and  $180^\circ$ .

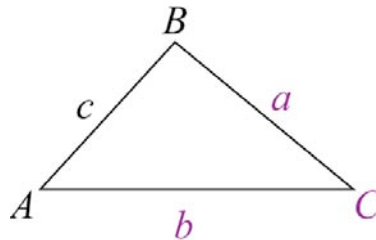
The  $180^\circ$  Rule, the Triangle Inequality, and the “Eating” Rule from [Notes 6.02](#) still apply.

**PART E: CASES**

Remember that the Law of Sines is applied in cases where you know two angles and one side (ASA and AAS) and in the ambiguous SSA case.

The Law of Cosines is applied in:

- The SSS case, in which all three sides are given, and
- The SAS case, in which two sides and their included angle are given. For example, we could be given side  $a$ , Angle  $C$ , and side  $b$  (see the figure below). This is actually the case for which the area of the triangle can be readily computed using the formula from [Section 6.1](#), whose proof was related to the proof for the Law of Sines.



The Law of Sines cannot handle these cases.

Can you see how these cases can yield either no triangle or exactly one?

Variations such as  $c^2 = a^2 + b^2 - 2ab\cos C$  are directly suited for the SAS case.

Variations such as  $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$  are directly suited for the SSS case.

**PART F: APPLYING THE LAW OF COSINES**

(See the Flowchart for Sections 6.1-6.2 on my website. It is better suited for practical use than for memorization.)

In the SSS case, you can quickly check to see if the Triangle Inequality is violated. Then, there are no triangles as solutions.

Strategies get a little complicated if you are going to use the Law of Sines in conjunction with the Law of Cosines ....

Technical Note: If you mix the two laws, you should find the largest angle (i.e., the angle “eating” the longest side) first in the SSS case. Then, you can eliminate any further candidates for obtuse angles (since a triangle can have at most one, and you have found the largest angle), and no SSA-style ambiguities arise.

In the SAS case, you should find the smaller angle of the two unknown angles (i.e., the one “eating” the shorter of the two given sides), because you know that that angle is acute; the remaining angle can then be obtained by subtraction via the  $180^\circ$  Rule.

You could, instead, use the Law of Cosines throughout, in which case there are no such ambiguities. (The computations may be a bit more involved, though, which is why some books prefer to mix in the Law of Sines.) As far as triangles go, although a sin value in the interval  $(0, 1)$  can yield two possible angles (an acute angle and its supplement), a cos value in the interval  $(-1, 1)$  can yield only one possible angle.

It is “game over” for possible solutions if you ever find that the cosine of an angle has to be outside the interval  $(-1, 1)$ .

If you are given a word problem, or if you are given a choice as to how to name the parts of the triangle at hand, you may want to label parts in a manner consistent with the first formula given for the Law of Cosines in [Notes 6.09](#).

**PART G: EXAMPLES**Example: SAS Case

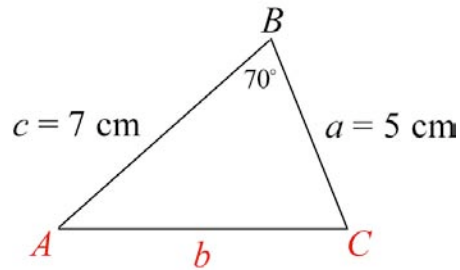
Given:  $a = 5$  cm,  $c = 7$  cm, and  $B = 70^\circ$ .

Solve the triangle. In your final answers, round off lengths to the nearest hundredth of a centimeter and angles to the nearest tenth of a degree.

Solution

Sketch a model triangle.

(Information yet to be determined is in red.)

Find Side  $b$ :

Use a variation of the Law of Cosines.

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$b = \sqrt{(5)^2 + (7)^2 - 2(5)(7)\cos 70^\circ} \quad (\text{Take the "+" root.})$$

$$\mathbf{b \approx 7.08 \text{ cm}}$$

Calculator Comments: You may want to work out the term with  $\cos 70^\circ$  first. Remember to incorporate the "-". Remember to take the square root at the end.

We will continue to use variations of the Law of Cosines instead of mixing in the Law of Sines, because the latter requires more strategizing about order. However, using the Law of Cosines may require more concentration on your part with respect to memorization, algebraic manipulations, and calculator computations.

Find Angle A:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

You can derive the above formula by starting with  
 $a^2 = b^2 + c^2 - 2bc \cos A$  and solving for  $\cos A$ .

$$\approx \frac{(7.08)^2 + (7)^2 - (5)^2}{2(7.08)(7)}$$

Calculator comments: If possible, try to keep your calculator's result for  $b$  in memory so that you may recall it. Also, process the numerator before dividing by the three factors in the denominator.

$$\approx 0.748$$

Calculator comments: Keep your calculator's result (not your rounded version) when you press the  $\cos^{-1}$  button. Make sure your calculator is in DEGREE mode, so that it automatically converts from radians (which really correspond to inverse trig values) to degrees.

Note: If we had obtained a cos value outside of the interval  $(-1, 1)$ , then no solution triangle would have existed.

$$A \approx 41.6^\circ$$

Find Angle C:

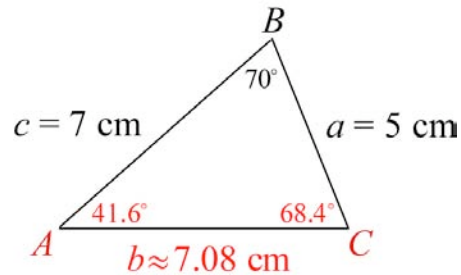
Use the 180° Rule. (It's a lot easier than the Law of Cosines.)

$$C \approx 180^\circ - 70^\circ - 41.6^\circ$$

$$C \approx 68.4^\circ$$

Use the “Eating” Rule to Check:

Make sure that larger angles “eat” longer sides.  
The relative side lengths seem appropriate, also.



Example: SSS Case

Given:  $a = 4$  in.,  $b = 5$  in.,  $c = 7$  in.

Solve the triangle. In your final answers, round off angles to the nearest tenth of a degree.

Solution

A model triangle may not be as helpful here.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(4)^2 + (5)^2 - (7)^2}{2(4)(5)} \Rightarrow C \approx 101.5^\circ$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(5)^2 + (7)^2 - (4)^2}{2(5)(7)} \Rightarrow A \approx 34.0^\circ$$

$$B \approx 180^\circ - 101.5^\circ - 34.0^\circ$$

$$B \approx 44.5^\circ$$

Observe that larger angles “eat” longer sides.  
(C “eats” c, the longest side, and so forth.)



**PART H: THE AREA OF A TRIANGLE: HERON'S FORMULA (SSS CASE)**

If you know the three sides of a triangle (if one exists), Heron's Formula can be used to quickly find its area.

Remember that, if you have SAS information, you can use the formulas in [Notes 6.07](#).

**Area of a Triangle: Heron's Formula (SSS Case)**

Let  $a$ ,  $b$ , and  $c$  be the three sides (i.e., side lengths) of a triangle.

Let the semiperimeter  $s = \frac{a + b + c}{2}$ . (This is half of the perimeter.)

Then, the area of the triangle is given by:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Observe that the formula is symmetric in  $a$ ,  $b$ , and  $c$  (so it does not matter which label goes with which side), and it does not require knowledge of any angles.

Note: If the sum of any two sides of the triangle does **not** exceed the third (i.e., if the Triangle Inequality falls apart), then the formula does **not** yield a positive real number. Let's say  $a + b \not\geq c$ . Then,  $s - c \leq 0$ . (It takes a little bit of work to see why.)

The proof on [p.470 of Larson](#) employs an area formula from [Notes 6.07](#) and the Law of Cosines.

See [p.422 of Larson](#) to see an Example and some info on Heron of Alexandria.