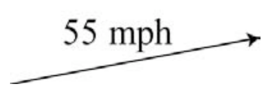
SECTION 6.3: VECTORS IN THE PLANE

Assume a, b, c, and d are real numbers.

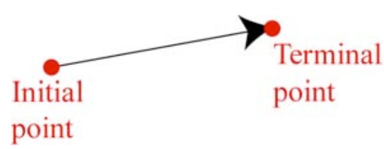
PART A: INTRO

A scalar has magnitude but not direction. We think of real numbers as scalars, even if they are negative. For example, a speed such as 55 mph is a scalar quantity.

A vector has both magnitude and direction. A vector **v** (written as $\vec{v}$ of $\vec{v}$ if you can't write in boldface) has magnitude $\|\mathbf{v}\|$. The length of a vector indicates its magnitude. For example, the directed line segment ("arrow") below is a velocity vector:



An equal vector (together with labeled parts) is shown below. Vectors with the same magnitude and direction (but not necessarily the same position) are equal.



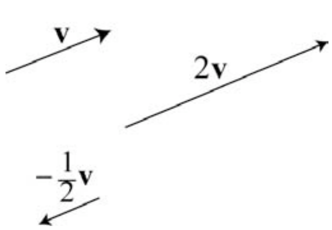
PART B: SCALAR MULTIPLICATION OF VECTORS

A scalar multiple of **v** is given by $c\mathbf{v}$, where c is some real scalar.

This new vector, $c\mathbf{v}$, is $|c|$ times as long as **v**.

If $c < 0$, then $c\mathbf{v}$ points in the opposite direction from the direction **v** points in.

Examples:



The vector $-\frac{1}{2}\mathbf{v}$ is referred to as "the opposite of $\frac{1}{2}\mathbf{v}$."

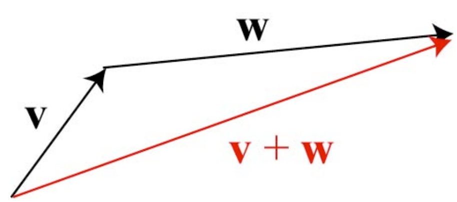
PART C: VECTOR ADDITION

Vector addition can correspond to combined (or net) effects.
For example, if $\mathbf{v}$ and $\mathbf{w}$ are force vectors, the resultant vector $\mathbf{v}+\mathbf{w}$ represents net force.

Vector subtraction may be defined as follows: $\mathbf{v}-\mathbf{w}=\mathbf{v}+\left(-\mathbf{w}\right)$.

There are two easy ways we can graphically represent vector addition:

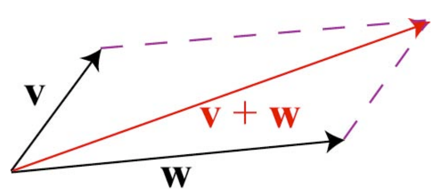
Triangle Law



To draw $\mathbf{v}+\mathbf{w}$, we place the tail of $\mathbf{w}$ at the head of $\mathbf{v}$, and we draw an arrow from the tail of $\mathbf{v}$ to the head of $\mathbf{w}$.

This may be better for representing sequential effects and displacements.

Parallelogram Law



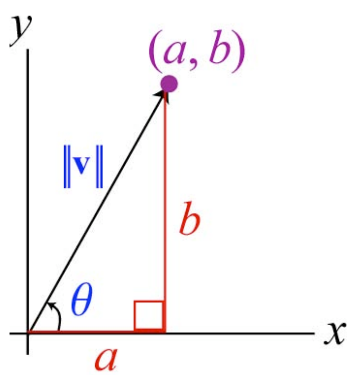
To draw $\mathbf{v}+\mathbf{w}$, we draw $\mathbf{v}$ and $\mathbf{w}$ so that they have the same initial point, we construct the parallelogram (if any) that they determine, and we draw an arrow from the common initial point to the opposing corner of the parallelogram.

This may be better for representing simultaneous effects and net force.

PART D: VECTORS IN THE RECTANGULAR (CARTESIAN) PLANE

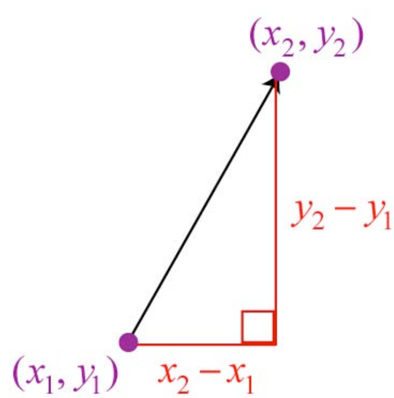
Let $\mathbf{v} = \langle a, b \rangle$. This is called the component form of v. We call a the horizontal component of **v**, b the vertical component, and the $\langle \; \rangle$ symbols angle brackets.

The position vector for **v** is drawn from the origin to the point (a, b).
It is the most convenient representation of **v**.

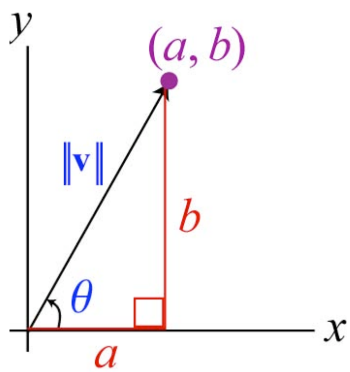


θ is a direction angle for **v**. We treat direction angles as standard angles here.
Remember that $\|\mathbf{v}\|$ is the magnitude, or length, of **v**.

A directed line segment drawn from the point (x_1, y_1) to the point (x_2, y_2) represents the vector $\langle x_2 - x_1, \; y_2 - y_1 \rangle$.



PART E: FORMULAS



If we are given a and b ...

By the Distance Formula (or the Pythagorean Theorem),

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2}$$

How can we relate θ, a, and b?

Choose θ such that:

$\tan\theta = \dfrac{b}{a}$ $(\text{if } a \neq 0)$, and
θ is in the correct Quadrant

Example

If $\mathbf{v} = \langle -3, 5 \rangle$, find $\|\mathbf{v}\|$ and θ, where $0 \le \theta < 360^\circ$.
Round off θ to the nearest tenth of a degree.

Solution

Find $\|\mathbf{v}\|$:

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-3)^2 + (5)^2} \\ &= \mathbf{\sqrt{34}} \end{aligned}$$

Find θ:

$$\begin{aligned} \tan\theta &= \frac{b}{a} \\ &= \frac{5}{-3} \\ &= -\frac{5}{3} \end{aligned}$$

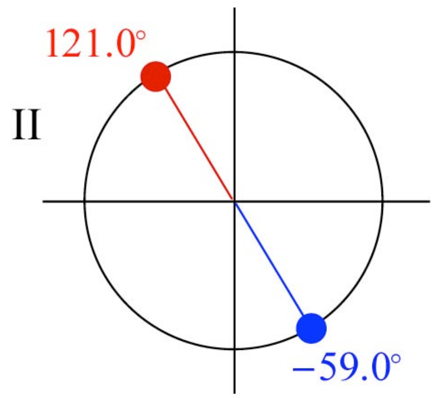
Warning: Make sure your calculator is in DEGREE mode when you press the $\tan^{-1}$ button.

Warning: The result may not be your answer. In fact, for this problem, it is not.

In degrees, $\tan^{-1}\left(-\frac{5}{3}\right) \approx -59.0^\circ$. However, this would be an inappropriate choice for θ, even without the restriction $0 \le \theta < 360^\circ$. This is because -59.0° is a Quadrant IV angle, whereas the point $(-3, 5)$ (and, therefore, the position vector for $\mathbf{v} = \langle -3, 5 \rangle$) is in Quadrant II.

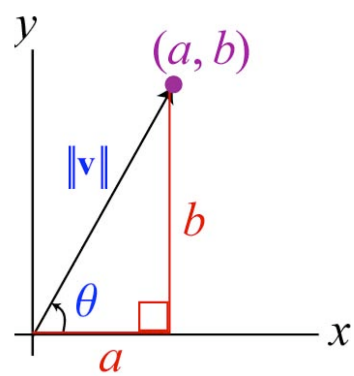
We require θ to be a Quadrant II angle in $\left[0^\circ, 360^\circ\right)$.
There is only one such angle:

$$\theta \approx -59.0^\circ + 180^\circ$$
$$\theta \approx \mathbf{121.0^\circ}$$



Answers: $\boxed{\|\mathbf{v}\| = \sqrt{\mathbf{34}},\ \boldsymbol{\theta} \approx \mathbf{121.0^\circ}}$

If we are given $\|\mathbf{v}\|$ and θ ...



$$\cos\theta = \frac{a}{\|\mathbf{v}\|} \qquad \sin\theta = \frac{b}{\|\mathbf{v}\|}$$
$$a = \|\mathbf{v}\|\cos\theta \qquad b = \|\mathbf{v}\|\sin\theta$$

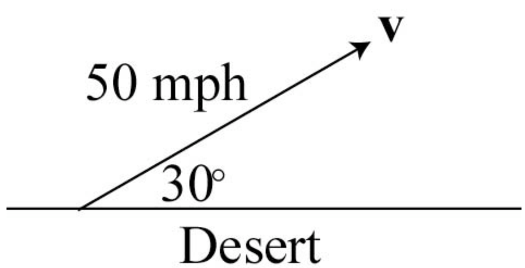
Therefore, $\boxed{\begin{aligned}\mathbf{v} &= \langle a, b\rangle \\ &= \left\langle \|\mathbf{v}\|\cos\theta,\ \|\mathbf{v}\|\sin\theta \right\rangle\end{aligned}}$

These formulas allow us to resolve a vector into its horizontal and vertical components.

Example

Out in the flat desert, a projectile is shot at a speed of 50 mph and an angle of elevation of 30°. Give the component form of the initial velocity vector **v**.

Solution



$\|\mathbf{v}\| = 50$ (mph), and $\theta = 30^\circ$.

$$\begin{aligned}\mathbf{v} &= \left\langle \|\mathbf{v}\|\cos\theta,\ \|\mathbf{v}\|\sin\theta \right\rangle \\ &= \left\langle 50\cos 30^\circ,\ 50\sin 30^\circ \right\rangle \\ &= \left\langle 50\left(\frac{\sqrt{3}}{2}\right),\ 50\left(\frac{1}{2}\right) \right\rangle \\ &== \left\langle 25\sqrt{3},\ 25 \right\rangle\end{aligned}$$

We now know the horizontal and vertical components of the initial velocity vector.

PART F: COMPUTATIONS WITH VECTORS

To add or subtract vectors, we add or subtract (in order) the components of the vectors.

$$\langle a, b\rangle + \langle c, d\rangle = \langle a+c,\ b+d\rangle$$
$$\langle a, b\rangle - \langle c, d\rangle = \langle a-c,\ b-d\rangle$$

To multiply a vector by a scalar, we multiply each component of the vector by the scalar.

$$c\langle a, b\rangle = \langle ca, cb\rangle$$

Example

If $\mathbf{v} = \langle 3, 5\rangle$ and $\mathbf{w} = \langle -1, -2\rangle$, find $4\mathbf{v} - 2\mathbf{w}$.

Solution

$$\begin{aligned} 4\mathbf{v} - 2\mathbf{w} &= 4\langle 3, 5\rangle - 2\langle -1, -2\rangle \\ &= \langle 12, 20\rangle + \langle 2, 4\rangle \qquad \text{(Adding is easier!)} \\ &= \langle 14, 24\rangle \end{aligned}$$

Although "scalar division" is a bit informal, we can define (if $c \neq 0$):

$$\frac{\langle a, b\rangle}{c} = \frac{1}{c}\langle a, b\rangle = \left\langle \frac{a}{c}, \frac{b}{c}\right\rangle$$

PART G: UNIT VECTORS

A unit vector has length (or magnitude) 1. Unit vectors are often denoted by **u**.

Given a vector **v**, the unit vector in the direction of **v** is given by:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \quad \left(\text{or } \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right)$$

It turns out that this normalization process is useful in Multivariable Calculus (Calculus III: Math 252 at Mesa) and Linear Algebra (Math 254 at Mesa).

Example

Find the unit vector in the direction of the vector **v**, if **v** can be represented by a directed line segment from $(1, 2)$ to $(4, 6)$.

Solution

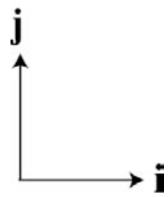
Find **v**: $\mathbf{v} = \langle 4 - 1,\ 6 - 2 \rangle = \langle 3, 4 \rangle$

Find its magnitude: $\| \mathbf{v} \| = \| \langle 3, 4 \rangle \| = \sqrt{(3)^2 + (4)^2} = 5$

The desired unit vector is: $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 3, 4 \rangle}{5} = \boxed{\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle}$

Standard Unit Vectors

$\mathbf{i} = \langle 1, 0 \rangle$, and
$\mathbf{j} = \langle 0, 1 \rangle$



These are often used in physics.

The vector $\langle a, b \rangle$ can be written as $a\mathbf{i} + b\mathbf{j}$, a linear combination of **i** and **j**.

For example, the answer in the previous Example, $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$, can be written as

$\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$.

Read the Historical Note on p.431 about Hamilton and Maxwell.