SECTION 6.4: VECTORS AND DOT PRODUCTS

Assume v_1, v_2, w_1, and w_2 are real numbers. For now, we will deal with vectors in the plane.

PART A: DOT PRODUCTS

How do we multiply vectors? There are two common types of products of vectors: the dot product (also known as the Euclidean inner product), and the cross product (also known as the vector product).

Dot Product (Algebraic Definition)

If $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$, then the dot product of **v** and **w** is given by:

$$\mathbf{v} \bullet \mathbf{w} = v_1 w_1 + v_2 w_2$$

In words, you add the products of corresponding components.
Dot products, themselves, are scalars.

Example

$$\begin{aligned}\langle 7, 3 \rangle \bullet \langle 2, -4 \rangle &= (7)(2) + (3)(-4) \\ &= 14 - 12 \\ &= 2\end{aligned}$$

PART B: PROPERTIES OF THE DOT PRODUCT

See p.440 in Larson. All but Property #4 are shared by the operation of multiplication of real numbers.

Property #1) The dot product is commutative: $\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$

Property #3) The dot product distributes over vector addition: $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

Property #4) Relating dot product and vector length: $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2$

Observe that both sides equal $v_1^2 + v_2^2$.

PART C: THE ANGLE BETWEEN TWO VECTORS

Given two nonzero vectors **v** and **w**, let θ be the angle between them (i.e., between their position vectors) that is in the interval $\left[0^\circ, 180^\circ\right]$.

$$\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

Think: The dot product over the product of the lengths.

The proof employs the Law of Cosines.

Note: Again, books avoid the $\cos^{-1}$ notation here, because angles between vectors are often given in degrees. The $\cos^{-1}$ button works nicely, though, because (if you are in DEGREE mode), you are guaranteed to get an angle in $\left[0^\circ, 180^\circ\right]$.

Note: Observe that the formula is symmetric in **v** and **w**, so it doesn't matter which way you name the vectors.

Technical Note: The Cauchy-Schwarz Inequality guarantees that the right-hand side of the formula is a value in $\left[-1, 1\right]$.

Two vectors **v** and **w** are orthogonal $\Leftrightarrow$ $\mathbf{v} \bullet \mathbf{w} = 0$.

This happens $\Leftrightarrow$ Either vector is **0**, the zero vector $\langle 0, 0 \rangle$ in the plane, or if $\theta = 90^\circ$. The terms "orthogonal" and "perpendicular" are frequently interchangeable.

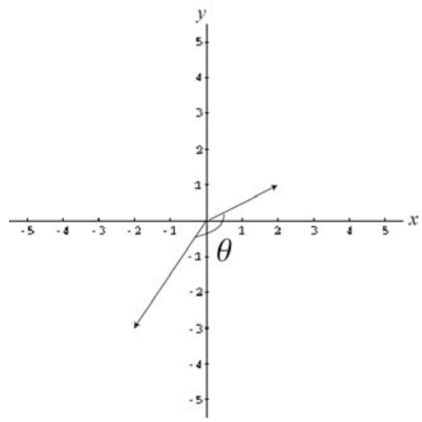
We say that **0** is orthogonal to every vector in the plane.

Example

Find the angle between $\mathbf{v} = \langle 2, 1 \rangle$ and $\mathbf{w} = \langle -2, -3 \rangle$ to the nearest tenth of a degree.

Solution

(Optional sketch:)



$$\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$
$$= \frac{\langle 2, 1 \rangle \bullet \langle -2, -3 \rangle}{\|\langle 2, 1 \rangle\|\|\langle -2, -3 \rangle\|}$$
$$= \frac{-4 - 3}{\sqrt{(2)^2 + (1)^2}\sqrt{(-2)^2 + (-3)^2}}$$
$$= \frac{-7}{\sqrt{5}\sqrt{13}}$$
$$= -\frac{7}{\sqrt{65}} \quad (\approx -0.868)$$

Make sure you are in DEGREE mode on your calculator.
Press the $\cos^{-1}$ button.

$$\theta = \cos^{-1}\left(-\frac{7}{\sqrt{65}}\right)$$
$$\theta \approx 150.3^\circ$$

PART D: WORK

The work W applied on an object by a constant force $\mathbf{F}$ as the object moves along the displacement vector $\mathbf{d}$ is given by: $W = \mathbf{F} \bullet \mathbf{d}$

Preliminaries: Dot Product (Geometric Definition)

From Notes 6.29, we had: $\cos\theta = \dfrac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$

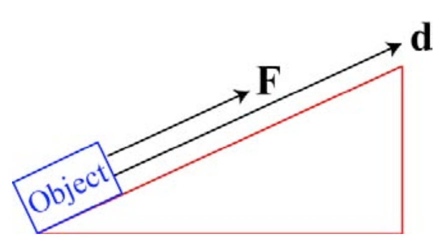
We can solve for $\mathbf{v} \bullet \mathbf{w}$ and obtain a geometric definition for the dot product:

$$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$$

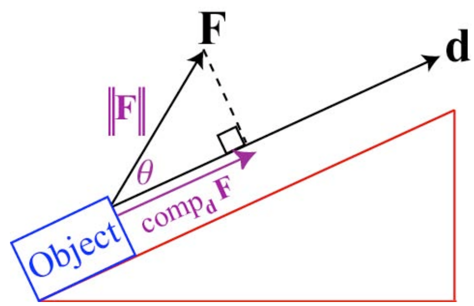
Reasoning Behind the Work Formula

If the force $\mathbf{F}$ being applied acts in the same direction as the direction of motion (i.e., the direction of $\mathbf{d}$), then the work done is given by:

$$\begin{aligned} W &= (\text{magnitude of force})(\text{distance traveled}) \\ &= \|\mathbf{F}\| \|\mathbf{d}\| \end{aligned}$$



In general, if **F** does not necessarily act in the same direction as **d**, then we replace the magnitude of **F**, denoted by $\|\mathbf{F}\|$, with the component of **F** in the direction of **d**, denoted by $\text{comp}_{\mathbf{d}}\mathbf{F}$. The latter represents the "relevant aspect" of the force in the direction of motion.



The work done is given by:

$$
\begin{aligned}
W &= \left(\text{comp}_{\mathbf{d}}\mathbf{F}\right)\left(\text{distance traveled}\right) \\
&= \left(\|\mathbf{F}\|\cos\theta\right)\left(\|\mathbf{d}\|\right) \\
&= \|\mathbf{F}\|\,\|\mathbf{d}\|\cos\theta \\
&= \mathbf{F}\bullet\mathbf{d}
\end{aligned}
$$

(by the Geometric Definition of Dot Product)

In Calculus: You will discuss the work applied on an object by a nonconstant (or variable) force field as the object moves along a nonlinear path, or even a path in three-dimensional space. You see this in Calculus III: Math 252 at Mesa.

Note: The ideas of this section extend naturally to three dimensions and (for the algebraic perspective on dot products) even higher dimensions.