

## SECTION 6.5: TRIG (AND EULER / EXPONENTIAL) FORMS OF A COMPLEX NUMBER

See the [Handout on my website](#).

### PART A: DIFFERENT FORMS OF A COMPLEX NUMBER

Let  $a$ ,  $b$ , and  $r$  be real numbers, and let  $\theta$  be measured in either degrees or radians (in which case it could be treated as a real number.)

We often let  $z$  denote a complex number.

Standard or Rectangular Form:  $z = a + bi$

We saw this form in [Section 2.4](#).

The complex number  $a + bi$  may be graphed in the complex plane as either the point  $(a, b)$  or as the position vector  $\langle a, b \rangle$ , in which case our analyses from [Section 6.3](#) become helpful.

Trig Form:  $z = r(\cos \theta + i \sin \theta)$

$r$  takes on the role of  $\|\mathbf{v}\|$  from our discussion of vectors. It is the distance of the point representing the complex number from 0.

$\theta$  has a role similar to the one it had in our discussion of vectors, namely as a direction angle, but this time in the complex plane.

This is derived from the Standard Form through the relations:

$$a = r \cos \theta, \text{ and } b = r \sin \theta$$

Recall from [Section 6.3](#), with  $r$  replacing  $\|\mathbf{v}\|$ :

$$r = \sqrt{a^2 + b^2}$$

Choose  $\theta$  such that:

$$\tan \theta = \frac{b}{a} \quad (\text{if } a \neq 0), \text{ and } \theta \text{ is in the correct Quadrant}$$

Euler (Exponential) Form:  $z = re^{i\theta}$

This is useful to derive various formulas in the [Handout](#).  
 In this class, you will not be required to use this form.

## **PART B: EXAMPLE**

### Example

Express  $z = 6 - 6i\sqrt{3}$  in Trig Form.

### Solution

Find  $r$ :

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(6)^2 + (-6\sqrt{3})^2} \\ &= \sqrt{36 + (36)(3)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

Find  $\theta$ :

$$\begin{aligned} \tan \theta &= \frac{b}{a} \\ &= \frac{-6\sqrt{3}}{6} \\ &= -\sqrt{3} \end{aligned}$$

Because  $z = 6 - 6i\sqrt{3}$ , we know that  $\theta$  must be in Quadrant IV.

An appropriate choice for  $\theta$  would be  $300^\circ$ , or  $\frac{5\pi}{3}$  radians.

Find the (really, “a”) Trig Form:

$$z = r(\cos \theta + i \sin \theta)$$

$$z = 12(\cos 300^\circ + i \sin 300^\circ) \quad \text{or} \quad z = 12\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$$

Note: In Euler (Exponential) Form, we have  $12e^{\frac{5\pi}{3}i}$ . We need a radian measure such as  $\frac{5\pi}{3}$  here.

### PART C: OPERATIONS ON COMPLEX NUMBERS USING TRIG FORM

Again, see the [Handout on my website](#).

When we **multiply** complex numbers (in Trig Form), we multiply their moduli, but we **add** their arguments; we go one step down in the order of operations when we deal with arguments. You can see this from the Euler (Exponential) Form.

When we **divide** complex numbers, we divide their moduli, but we **subtract** their arguments.

When we **raise** a complex number **to a power**, we raise the modulus to that power, but we **multiply** the argument by that power.

#### Examples

$$\text{Let } z_1 = 3(\cos 50^\circ + i \sin 50^\circ), \text{ and } z_2 = 4(\cos 10^\circ + i \sin 10^\circ).$$

Then (in Trig Form):

$$z_1 z_2 = 12(\cos 60^\circ + i \sin 60^\circ) \quad \leftarrow \text{This is } 6 + 6i\sqrt{3} \text{ in Rectangular Form.}$$

$$\frac{z_1}{z_2} = \frac{3}{4}(\cos 40^\circ + i \sin 40^\circ)$$

$$\begin{aligned} (z_1)^4 &= (3)^4 \cdot (\cos(4 \cdot 50^\circ) + i \sin(4 \cdot 50^\circ)) \\ &= 81(\cos 200^\circ + i \sin 200^\circ) \end{aligned}$$

Think About It: What happens to a point representing a complex number in the complex plane when we multiply the number by  $i$ ? How does this relate to our discussion on the powers of  $i$  back in [Section 2.4](#)?

## PART D: ROOTS OF A COMPLEX NUMBER

This is the most complicated story we have in this section.

We say:  $\sqrt{9} = 3$ , because 3 is the principal square root of 9. This is because 3 is the **nonnegative** square root of 9. You could also say that  $-2$  is the principal cube root of  $-8$ , because  $-2$  is the only **real** cube root of  $-8$ .

In this section, however, we will consider both 3 and  $-3$  as square roots of 9, because the square of both numbers is 9. (In fact, they are the only complex numbers whose square is 9.)

More generally, a nonzero complex number has  $n$  complex  $n^{\text{th}}$  roots, where  $n$  is a counting number (i.e., a positive integer) greater than 1.

### Example

Find the fourth (complex) roots of  $16i$ .

In other words, find all complex solutions of  $x^4 = 16i$ .

### Solution

The modulus of  $16i$  is:  $r = 16$ . We will call this “old  $r$ .”

All four fourth roots will have modulus: new  $r = \sqrt[4]{\text{old } r} = \sqrt[4]{16} = 2$

The argument of  $16i$  is:  $\theta = 90^\circ$ . We will call this “old  $\theta$ .”

According to the [Handout](#), the arguments of the roots are given by:

$$\frac{\theta + 2\pi k}{n} \quad \left( \text{or } \frac{\theta}{n} + \frac{2\pi}{n} k \right), \quad k = 0, 1, 2, \dots, n-1, \text{ where } \theta \text{ is any}$$

suitable argument in radians. The degree version looks like:

$$\frac{\theta + (360^\circ)k}{n} \quad \left( \text{or } \frac{\theta}{n} + \frac{360^\circ}{n} k \right), \quad k = 0, 1, 2, \dots, n-1$$

The forms in the parentheses present an easier approach:

Take  $\frac{\theta}{n}$  as the argument of one of the roots. Here, this would be:

$$\frac{90^\circ}{4} = 22.5^\circ, \text{ or } \frac{\pi/2}{4} = \frac{\pi}{8} \text{ in radians.}$$

The roots will be regularly spaced about the circle of radius 2 centered at 0. The “period” for the roots will be:

$$\frac{360^\circ}{n} = \frac{360^\circ}{4} = 90^\circ, \text{ or } \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ in radians.}$$

Let's deal with degrees in this problem.

Remember, all four fourth roots have modulus 2.

The following arguments are appropriate for the four roots:

$$22.5^\circ \xrightarrow{+90^\circ} 112.5^\circ \xrightarrow{+90^\circ} 202.5^\circ \xrightarrow{+90^\circ} 292.5^\circ$$

If we were to add  $90^\circ$  to the last argument, we would get an angle coterminal with  $22.5^\circ$ , and we do not take that as another root. Observe that we would get a Trig Form corresponding to the same Standard Form as the first root; we are looking for four **distinct** roots.

Trig Form for the roots:

Note: There are many acceptable possibilities for the arguments.

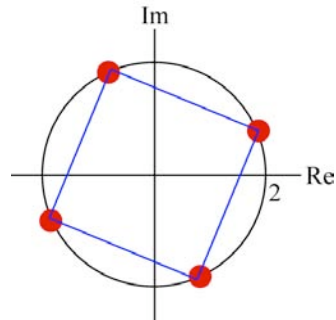
$$z_0 = 2(\cos 22.5^\circ + i \sin 22.5^\circ)$$

$$z_1 = 2(\cos 112.5^\circ + i \sin 112.5^\circ)$$

$$z_2 = 2(\cos 202.5^\circ + i \sin 202.5^\circ)$$

$$z_3 = 2(\cos 292.5^\circ + i \sin 292.5^\circ)$$

Here is a graph of the four roots (represented by points in red):



This should remind you of solving trig equations with multiple angles back in [Section 5.3](#).

In general, the  $n^{\text{th}}$  roots of a nonzero complex number correspond to the vertices of a regular  $n$ -gon. They exhibit the nice symmetry around a circle that we found in [Section 5.3](#) with trig equations with multiple angles.