SECTION 6.5: TRIG (AND EULER / EXPONENTIAL) FORMS OF A COMPLEX NUMBER

See the Handout on my website.

PART A: DIFFERENT FORMS OF A COMPLEX NUMBER

Let a, b, and r be real numbers, and let θ be measured in either degrees or radians (in which case it could be treated as a real number.)

We often let z denote a complex number.

Standard or Rectangular Form: $z = a + bi$

We saw this form in Section 2.4.

The complex number $a + bi$ may be graphed in the complex plane as either the point (a, b) or as the position vector $\langle a, b \rangle$, in which case our analyses from Section 6.3 become helpful.

Trig Form: $z = r(\cos\theta + i\sin\theta)$

r takes on the role of $\|\mathbf{v}\|$ from our discussion of vectors. It is the distance of the point representing the complex number from 0.

θ has a role similar to the one it had in our discussion of vectors, namely as a direction angle, but this time in the complex plane.

This is derived from the Standard Form through the relations:

$$a = r\cos\theta, \text{ and } b = r\sin\theta$$

Recall from Section 6.3, with r replacing $\|\mathbf{v}\|$:

$$r = \sqrt{a^2 + b^2}$$

Choose θ such that:

$$\tan\theta = \frac{b}{a} \quad (\text{if } a \neq 0), \text{ and } \theta \text{ is in the correct Quadrant}$$

Euler (Exponential) Form: $z = re^{i\theta}$

This is useful to derive various formulas in the Handout.
In this class, you will not be required to use this form.

PART B: EXAMPLE

Example

Express $z = 6 - 6i\sqrt{3}$ in Trig Form.

Solution

Find r:

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(6)^2 + \left(-6\sqrt{3}\right)^2} \\ &= \sqrt{36 + (36)(3)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

Find θ:

$$\begin{aligned} \tan\theta &= \frac{b}{a} \\ &= \frac{-6\sqrt{3}}{6} \\ &= -\sqrt{3} \end{aligned}$$

Because $z = 6 - 6i\sqrt{3}$, we know that θ must be in Quadrant IV.

An appropriate choice for θ would be 300°, or $\frac{5\pi}{3}$ radians.

Find the (really, "a") Trig Form:

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 12(\cos 300^\circ + i\sin 300^\circ) \quad \text{or} \quad z = 12\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Note: In Euler (Exponential) Form, we have $12e^{\frac{5\pi}{3}i}$. We need a radian measure such as $\frac{5\pi}{3}$ here.

PART C: OPERATIONS ON COMPLEX NUMBERS USING TRIG FORM

Again, see the Handout on my website.

When we **multiply** complex numbers (in Trig Form), we multiply their moduli, but we **add** their arguments; we go one step down in the order of operations when we deal with arguments. You can see this from the Euler (Exponential) Form.

When we **divide** complex numbers, we divide their moduli, but we **subtract** their arguments.

When we **raise** a complex number **to a power**, we raise the modulus to that power, but we **multiply** the argument by that power.

Examples

Let $z_1 = 3(\cos 50^\circ + i\sin 50^\circ)$, and $z_2 = 4(\cos 10^\circ + i\sin 10^\circ)$.

Then (in Trig Form):

$$z_1 z_2 = 12(\cos 60^\circ + i\sin 60^\circ) \quad \leftarrow \text{This is } 6 + 6i\sqrt{3} \text{ in Rectangular Form.}$$

$$\frac{z_1}{z_2} = \frac{3}{4}(\cos 40^\circ + i\sin 40^\circ)$$

$$(z_1)^4 = (3)^4 \cdot (\cos(4 \cdot 50^\circ) + i\sin(4 \cdot 50^\circ))$$

$$= 81(\cos 200^\circ + i\sin 200^\circ)$$

Think About It: What happens to a point representing a complex number in the complex plane when we multiply the number by i? How does this relate to our discussion on the powers of i back in Section 2.4?

PART D: ROOTS OF A COMPLEX NUMBER

This is the most complicated story we have in this section.

We say: $\sqrt{9} = 3$, because 3 is the principal square root of 9. This is because 3 is the **nonnegative** square root of 9. You could also say that -2 is the principal cube root of -8, because -2 is the only **real** cube root of -8.

In this section, however, we will consider both 3 and -3 as square roots of 9, because the square of both numbers is 9. (In fact, they are the only complex numbers whose square is 9.)

More generally, a nonzero complex number has n complex n^{th} roots, where n is a counting number (i.e., a positive integer) greater than 1.

Example

Find the fourth (complex) roots of $16i$.
In other words, find all complex solutions of $x^4 = 16i$.

Solution

The modulus of $16i$ is: $r = 16$. We will call this "old r."

All four fourth roots will have modulus: new $r = \sqrt[4]{\text{old } r} = \sqrt[4]{16} = 2$

The argument of $16i$ is: $\theta = 90^\circ$. We will call this "old θ."

According to the Handout, the arguments of the roots are given by:
$\dfrac{\theta + 2\pi k}{n}$ $\left(\text{or } \dfrac{\theta}{n} + \dfrac{2\pi}{n} k \right)$, $k = 0, 1, 2, ..., n-1$, where θ is any suitable argument in radians. The degree version looks like:
$$\frac{\theta + \left(360^\circ\right)k}{n} \quad \left(\text{or } \frac{\theta}{n} + \frac{360^\circ}{n} k \right), \; k = 0, 1, 2, ..., n-1$$

The forms in the parentheses present an easier approach:

Take $\frac{\theta}{n}$ as the argument of one of the roots. Here, this would be:

$$\frac{90^\circ}{4} = 22.5^\circ \text{, or } \frac{\pi/2}{4} = \frac{\pi}{8} \text{ in radians.}$$

The roots will be regularly spaced about the circle of radius 2 centered at 0. The "period" for the roots will be:

$$\frac{360^\circ}{n} = \frac{360^\circ}{4} = 90^\circ \text{, or } \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ in radians.}$$

Let's deal with degrees in this problem.

Remember, all four fourth roots have modulus 2.

The following arguments are appropriate for the four roots:

$$22.5^\circ \xrightarrow{+90^\circ} 112.5^\circ \xrightarrow{+90^\circ} 202.5^\circ \xrightarrow{+90^\circ} 292.5^\circ$$

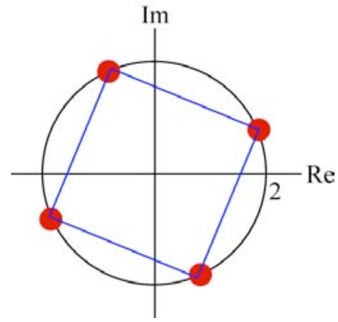
If we were to add 90° to the last argument, we would get an angle coterminal with 22.5°, and we do not take that as another root. Observe that we would get a Trig Form corresponding to the same Standard Form as the first root; we are looking for four **distinct** roots.

Trig Form for the roots:

<u>Note</u>: There are many acceptable possibilities for the arguments.

$$z_0 = 2\left(\cos 22.5^\circ + i\sin 22.5^\circ\right)$$
$$z_1 = 2\left(\cos 112.5^\circ + i\sin 112.5^\circ\right)$$
$$z_2 = 2\left(\cos 202.5^\circ + i\sin 202.5^\circ\right)$$
$$z_3 = 2\left(\cos 292.5^\circ + i\sin 292.5^\circ\right)$$

Here is a graph of the four roots (represented by points in red):



This should remind you of solving trig equations with multiple angles back in Section 5.3.

In general, the n^{th} roots of a nonzero complex number correspond to the vertices of a regular n-gon. They exhibit the nice symmetry around a circle that we found in Section 5.3 with trig equations with multiple angles.