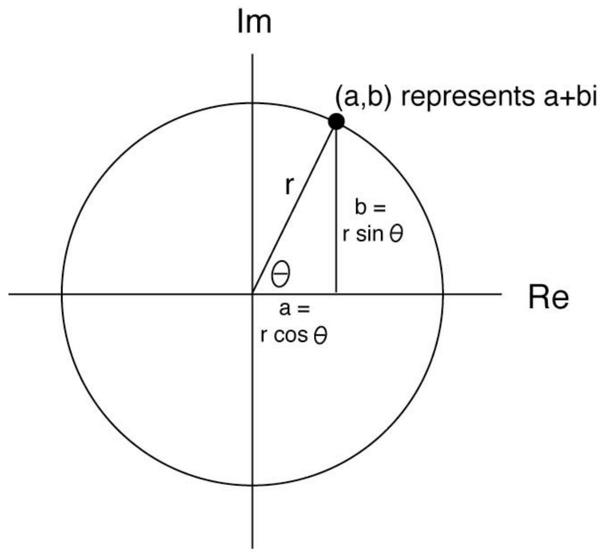
SECTION 6.5: TRIG (and EULER) FORMS OF A COMPLEX NUMBER

A Picture

Let $z = a + bi$. This is plotted as (a, b) in the complex plane.



r (modulus)

The absolute value (or modulus, plural moduli) of z is

$$\begin{aligned} r &= |a + bi| \\ &= \sqrt{a^2 + b^2} \end{aligned}$$

If z is a real number, then $b = 0$ and $r = |a|$, which is consistent with the notation for the absolute value of a real number.

θ (argument)

θ is an argument of z in the picture.
(Remember that infinitely many coterminal angles can be the argument.)
θ can be anything real if $z = 0$.

Finding θ:

$\tan\theta = \dfrac{b}{a}$ (Which quadrant of the complex plane does z lie in?)
(Maybe this is undefined.)

Trig (or "Polar") Form of a Complex Number

$z = \underbrace{(r\cos\theta)}_{a} + \underbrace{(r\sin\theta)}_{b}i$ or, more simply,

$z = r(\cos\theta + i\sin\theta)$

Euler Form of a Complex Number

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Famous Case

If $\theta = \pi$, we get

$$e^{i\pi} = \cos\pi + i\underbrace{\sin\pi}_{0}$$

$$e^{\pi i} = -1 \quad \leftarrow \text{We have } e^{\text{something}} = \text{(a negative number)!!!}$$

$$e^{\pi i} + 1 = 0$$

This last formula relates five of the most basic constants in mathematics: $e, \pi, i, 1,$ and 0 !!!

From Trig Form to Euler Form

$$z = r\underbrace{(\cos\theta + i\sin\theta)}_{e^{i\theta}} \quad \leftarrow \text{Trig (Polar) Form}$$

$$z = re^{i\theta} \quad \leftarrow \text{Euler Form}$$

Euler Form may be convenient when performing operations on complex numbers and when deriving related properties. Let's see…. (The textbook has different approaches.)

Multiplying Complex Numbers in Trig Form

Multiply the moduli, and add the arguments.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ i.e., $z_1 = r_1 e^{i\theta_1}$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ i.e., $z_2 = r_2 e^{i\theta_2}$

Then, $z_1 z_2 = r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]$ i.e., $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

Derivation using Euler Form

Why do we add the arguments instead of multiplying them?
When we multiply powers of e, we add the exponents.

$$\begin{aligned} z_1 z_2 &= \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) \\ &= r_1 r_2 e^{i\theta_1 + i\theta_2} \\ &= \underbrace{r_1 r_2}_{\text{new } r}\, e^{i\overbrace{(\theta_1 + \theta_2)}^{\text{new } \theta}} \\ &= r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right] \end{aligned}$$

Dividing Complex Numbers in Trig Form

Divide the moduli, and subtract the arguments.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ i.e., $z_1 = r_1 e^{i\theta_1}$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \neq 0$ i.e., $z_2 = r_2 e^{i\theta_2} \neq 0$

Then, $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$ i.e., $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Derivation using Euler Form

Why do we subtract the arguments instead of dividing them?
When we divide powers of *e*, we subtract the exponents.

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \\
&= \frac{r_1}{r_2} e^{i\theta_1 - i\theta_2} \\
&= \underbrace{\frac{r_1}{r_2}}_{\text{new } r} e^{i\overbrace{(\theta_1 - \theta_2)}^{\text{new } \theta}} \\
&= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]
\end{aligned}$$

Taking the *n*th Power of a Complex Number in Trig Form (*n* is a positive integer)

Take the *nth power* of the modulus, and multiply the argument by *n*.

If $z = r(\cos\theta + i\sin\theta)$ i.e., $z = re^{i\theta}$

Then, $\underbrace{z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]}_{\text{DeMoivre's Theorem}}$ i.e., $z^n = r^n e^{i(n\theta)}$

Derivation using Euler Form

When we raise a power of *e* to a power, we multiply the exponents.

$$\begin{aligned}
z &= re^{i\theta} \\
z^n &= \left(re^{i\theta}\right)^n \\
&= r^n e^{i\theta n} \\
&= \underbrace{r^n}_{\text{new } r} e^{i\overbrace{(n\theta)}^{\text{new } \theta}} \\
&= r^n\left[\cos(n\theta) + i\sin(n\theta)\right]
\end{aligned}$$

Finding *n*th Roots of a Complex Number in Trig Form (*n* is a positive integer)

Take the nth root of the modulus, and divide *n* consecutive versions of the argument by *n*.

If $z = r(\cos\theta + i\sin\theta)$ i.e., $z = re^{i\theta}$

Then, the *n* distinct *n*th roots of *z* (provided $z \neq 0$) are given by:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \quad \text{i.e.,} \quad z_k = \sqrt[n]{r}e^{i\left(\frac{\theta + 2\pi k}{n}\right)}$$

where $k = 0, 1, 2, ..., n-1$.

The *n*th power of each of these is *z*.

(This should remind you of solving trig equations involving multiples of angles.)

Derivation using Euler Form

When we take the nth root of a power of *e*, we divide the exponent by *n*.

$$z = re^{i(\theta + 2\pi k)}, \; k = 0, 1, 2, ..., n-1$$

$$z_k = \left[re^{i(\theta + 2\pi k)}\right]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}} e^{\frac{i(\theta + 2\pi k)}{n}}$$

$$= \underbrace{\sqrt[n]{r}}_{\text{new } r} e^{i\overbrace{\left(\frac{\theta + 2\pi k}{n}\right)}^{\text{new } \theta}}$$

$$= \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

Summary

Trig (Polar) Form: $z = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}$ (and consider the Quadrant that *z* lies in)

Euler Form: $z = re^{i\theta}$ (good for deriving properties)

	new r	new θ
$z_1 z_2$	$r_1 r_2$	$\theta_1 + \theta_2$
$\frac{z_1}{z_2}$	$\frac{r_1}{r_2}$	$\theta_1 - \theta_2$
z^n (DeMoivre)	r^n	$n\theta$
Roots	$\sqrt[n]{r}$	$\frac{\theta + 2\pi k}{n}$, $k = 0, 1, 2, ..., n-1$