CHAPTER 7: SYSTEMS AND INEQUALITIES

SECTIONS 7.1-7.3: SYSTEMS OF EQUATIONS

PART A: INTRO

A solution to a system of equations must satisfy **all** of the equations in the system.

In your Algebra courses, you should have learned methods for solving systems of linear equations, such as:

$$\begin{cases} A + B = 1 \\ A - 4B = 11 \end{cases}$$

We will solve this system using both the Substitution Method and the Addition / Elimination Method in Section 7.4 on Partial Fractions.

In some cases, these methods can be extended to nonlinear systems, in which at least one of the equations is nonlinear.

PART B: THE SUBSTITUTION METHOD

See Example 1 on p.497.

Example (#8 on p.503)

Solve the nonlinear system: $\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$

Solution

We can, for example,

(Step 1) Solve the second equation for y in terms of x and then

(Step 2) Perform a substitution into the first equation.

$$\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases} \Rightarrow \underbrace{y = 2 - x^3}_{\text{Call this } star.} \nearrow \quad \Rightarrow 3x + \left(2 - x^3\right) = 2$$

$$3x + \cancel{2} - x^3 = \cancel{2}_0$$

$$3x - x^3 = 0$$

We may prefer to rewrite this last equation so that the nonzero side has a positive leading coefficient. We're more used to that setup.

$$0 = x^3 - 3x$$

Step 3) Solve $0 = x^3 - 3x$ for x.

Warning: Remember that dividing both sides by x is risky. We may lose solutions. We prefer the Factoring method.

$$0 = x\left(x^2 - 3\right)$$

You could factor $\left(x^2 - 3\right)$ over **R** or stop factoring here.

Apply the ZFP (Zero Factor Property):

$$x = 0 \quad \text{or} \quad \begin{aligned} x^2 - 3 &= 0 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

Warning: We're not done yet! We need to find the corresponding *y*-values.

Step 4) Back-substitute into *star*.

Observe that: $\left(\sqrt{3}\right)^3 = \left(\sqrt{3}\right)\left(\sqrt{3}\right)\left(\sqrt{3}\right) = 3\sqrt{3}$

x	$y = 2 - x^3$
0	$2 - (0)^3 = 2$
$\sqrt{3}$	$2 - \left(\sqrt{3}\right)^3 = 2 - 3\sqrt{3}$
$-\sqrt{3}$	$2 - \left(-\sqrt{3}\right)^3 = 2 - \left(-3\sqrt{3}\right) = 2 + 3\sqrt{3}$

Step 5) Write the solution set.

This is usually required if you are solving a system of equations.

Warning: Make sure that your solutions are written in the form (x, y), not (y, x).

The solution set here is:

$$\left\{(0,2),\ \left(\sqrt{3}, 2 - 3\sqrt{3}\right),\ \left(-\sqrt{3}, 2 + 3\sqrt{3}\right)\right\}$$

This consists of three real solutions written as ordered pairs.
We assume that ordered pairs are appropriate, since no mention is made of *z* or other variables.

Step 6) Check your solutions in the given system. (Optional)

Warning: There is a danger in trying to check solutions in a later equivalent system that you have written down, because you may have made an error by that point. Use the original system, before you got your dirty hands all over it!

For example, we can check the solution $(0,2)$ in the given system:

$$\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$$

Remember that a solution to a system of equations must satisfy **all** of the equations in the system.

$$\begin{cases} 3(0)+(2)=2 & \Rightarrow \quad 2=2 \\ (0)^3 - 2 + (2) = 0 & \Rightarrow \quad 0=0 \end{cases}$$

The solution $(0,2)$ checks out.

PART C: THE GRAPHICAL METHOD

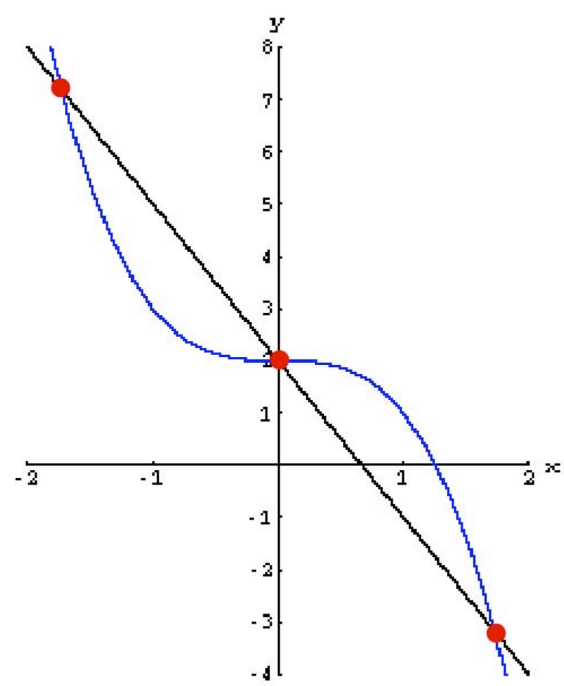
The Graphical Method for solving a system of equations requires that we graph all of the equations and then find the resulting intersection points common to **all** the graphs, if any. These points correspond to the **real** solutions to the system.

Warning: Books neglect to mention that this method is highly unreliable. For example, how can we tell visually if an intersection point is at $(0,2)$ as opposed to, say, $(0.01, 1.98)$? Although many textbook problems are designed to have "nice" solutions, we can't always assume that our solutions will only consist of integer coordinates.

Recall our Example from Part B. The solution set for the system

$$\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$$

was $\left\{(0,2),\ \left(\sqrt{3}, 2-3\sqrt{3}\right),\ \left(-\sqrt{3}, 2+3\sqrt{3}\right)\right\}$. Consider the graphs of the two equations in the system. Based only on the figure given for #8 on p.503 (or the one below), could you have obtained the last two solutions exactly from mere visual inspection?



Note: $\left(\sqrt{3}, 2-3\sqrt{3}\right) \approx (1.7, -3.2)$, and $\left(-\sqrt{3}, 2+3\sqrt{3}\right) \approx (-1.7, 7.2)$

The figure is helpful, however, in that it seems to confirm that the system has three real solutions (corresponding to the three red intersection points), and (with the help of a calculator) the three solutions we found seem to roughly check out graphically, at least up to the limits of our vision and the precision of the figure.

The graph in black is the graph of $3x + y = 2$, which can be rewritten as $y = -3x + 2$.
The graph in blue is the graph of $x^3 - 2 + y = 0$, which can be rewritten as $y = -x^3 + 2$.

You can roughly sketch these graphs by hand, but it may be difficult to accurately locate intersection points.

Example

Find all real solutions of the system: $\begin{cases} \frac{1}{2}y = \frac{1}{2}x + 1 \\ x^2 + y^2 = 1 \end{cases}$

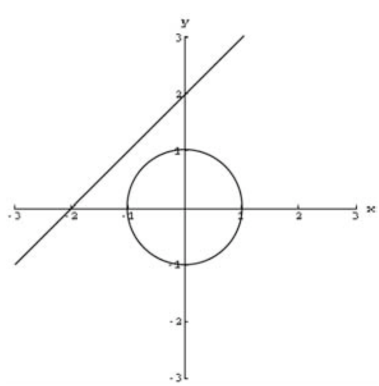
Solution

Let's multiply both sides of the first equation "through" by 2.
(i.e., We multiply each term on both sides by 2.) We obtain:

$$\begin{cases} y = x + 2 \\ x^2 + y^2 = 1 \end{cases}$$

Method 1 (Graphical Method)

The first equation gives us the line below.
The second equation gives us the unit circle below.



There are no intersection points, so there are no real solutions to the system. The solution set is: $\varnothing$, the empty or null set.

Method 2 (Substitution Method)

$$\begin{cases} y = x + 2 \\ x^2 + y^2 = 1 \end{cases}$$

The first equation is already solved for y in terms of x.
Let's substitute into the second equation.

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + (x+2)^2 &= 1 \\ x^2 + x^2 + 4x + 4 &= 1 \\ 2x^2 + 4x + 4 &= 1 \\ 2x^2 + 4x + 3 &= 0 \end{aligned}$$

According to the QF (Quadratic Formula), the complex solutions of this equation are: $\dfrac{-4 \pm \sqrt{-8}}{4}$, which simplify to $\dfrac{-2 \pm i\sqrt{2}}{2}$, or $-1 \pm \dfrac{\sqrt{2}}{2} i$.

There are no viable **real** values for x, so the system has no real solutions, and the solution set is $\varnothing$.

This confirms our findings from Method 1.

Note: It is not necessary to work out the entire QF to conclude that there are no viable real values for x. It is sufficient to observe that the discriminant is negative: $b^2 - 4ac = (4)^2 - 4(2)(3) = -8 < 0$
In other problems, the discriminant can be used in conjunction with the Test for Factorability described in Section 1.5: Notes 1.48 to see if our equation can be solved quickly.

In Precalculus, a system of equations with no real solutions (i.e., an empty solution set) is called inconsistent in the sense that there is no real solution that consistently solves all of the equations in the system. The equations cannot be reconciled.

PART D: THE ADDITION / ELIMINATION METHOD

This method is based on the principle that, when you add equals to equals, you get equals. See Examples 1-3 on pp.507-509.

Example

Solve the nonlinear system: $\begin{cases} 2x+5y^2=35 \\ 7x+2y^2=14 \end{cases}$

Solution

The Substitution Method may get messy if we try to solve for x or y^2 in either equation.

The Addition Method may not be helpful to solve some nonlinear systems, but it is helpful here.

We can easily eliminate the y^2 terms, for example, by:

1) Multiplying "through" both sides of the first equation by 2,
2) Multiplying "through" both sides of the second equation by (-5),

and

3) Adding equals to equals to obtain a third equation that we can use to crack the system. We will informally refer to this process as adding equations.

$$\begin{cases} 2x+5y^2=35 & \leftarrow \cdot(2) \\ 7x+2y^2=14 & \leftarrow \cdot(-5) \end{cases}$$

In our new equivalent system, the coefficients of the y^2 terms will be opposites, and (when adding the equations) we will be able to eliminate those terms and then solve for x.

$$\begin{cases} 4x+10y^2 = 70 \\ -35x-10y^2=-70 \end{cases}$$
$$\begin{aligned} -31x &= 0 \\ x &= 0 \end{aligned}$$

Warning 1: Remember to multiply the **right-hand sides** of the given equations by 2 and -5, respectively. People often focus on the left-hand sides so much that they forget about the right-hand sides.

Warning 2: Why didn't we multiply the second equation by 5 (instead of -5) and then **subtract** equals from equals? Although that would have been a correct procedure, people often make mechanical errors when subtracting. We generally prefer to **add**, instead, even if that means that we multiply both sides of an equation by a negative number. One possible exception is given in Notes 7.11. This issue will come up later in Chapter 8, when we study matrices.

Warning 3: We were lucky that the right-hand side of the resulting equation is 0, but it doesn't always have to be 0. What if you had eliminated the x terms, instead?

We can now "plug in" $x = 0$ into any of the four equations at the bottom of Notes 7.08 that contain both x and y. If we plug into the first given equation:

$$\begin{aligned} 2x + 5y^2 &= 35 \\ 2(0) + 5y^2 &= 35 \\ 5y^2 &= 35 \\ y^2 &= 7 \\ y &= \pm\sqrt{7} \end{aligned}$$

For our solutions, we require that $x = 0$, and then y can either be $\sqrt{7}$ or $-\sqrt{7}$.

Warning: Make sure you know which values of y correspond to which values of x. In Multivariable Calculus (Calculus III: Math 252 at Mesa), this issue will arise in a big way when you study Lagrange Multipliers and optimization.

Our solution set is: $\left\{\left(0, \sqrt{7}\right), \left(0, -\sqrt{7}\right)\right\}$.

Some people write $\left(0, \pm\sqrt{7}\right)$, though that may be ambiguous.

See Warning 1 in Section 1.5: Notes 1.46.

PART E: HOW MANY SOLUTIONS CAN A SYSTEM OF EQUATIONS HAVE?

We have seen nonlinear systems of equations with 0, 2, and 3 solutions. In fact, **nonlinear** systems can potentially have any whole number of solutions; they can even have infinitely many solutions. Consider the system:

$$\begin{cases} y = \sin x \\ y = 0 \end{cases}$$

However, the only possibilities for a system of **linear** equations are: 0, 1, or infinitely many solutions. Consider systems of two linear equations in two unknowns (say x and y) – and the graphs of those equations in the xy-plane. Remember that real solutions correspond to intersection points.

How many solutions?	Example	System is …
0	 different parallel lines	Inconsistent
1	 non-parallel lines	Consistent
Infinitely many	 coincident lines	Consistent

Warning: The term dependence seems to have different meanings in Introductory Algebra books and in Linear Algebra books. We will ignore this issue.

PART F: SPECIAL CASES

Example

Solve the system $\begin{cases} x + y = 2 \\ x + y = 1 \end{cases}$

Solution

Here, it is easy to "subtract" the second equation from the first.
We obtain $0 = 1$, which cannot be satisfied by any ordered pair (x, y).
In other words, there is no (x, y) for which $0 = 1$ is true.
Therefore, the system has no solutions, and the solution set is $\varnothing$.

Technical Logic Note: If this system had a solution (x, y), then $0 = 1$ would have to be true. However, we know that $0 = 1$ is not true. Therefore, the system has no solution. This is an example of indirect reasoning, which is based on the logical equivalence between an if-then statement and its contrapositive. (See Notes P.06 to P.08.)

Note: We can also say that, because $x + y = x + y$, we require that $2 = 1$ be true for any solution to the system.

We will discuss systems of linear equations with infinitely many solutions at the end of Section 8.1.

Example

How many solutions does the following system have?

$$\begin{cases} y = x + 2 \\ y = x + 2 \\ 0 = 1 \end{cases}$$

Solution

If you "subtract" the second equation from the first, then you will obtain $0 = 0$.

Warning: You may then be tempted to conclude that this system has infinitely many solutions, especially since the first two equations represent the same line in the *xy*-plane. You would be wrong!

The equation $0 = 1$ has no solutions; there is no ordered pair (x, y) such that $0 = 1$ can be satisfied. (That equation ruins it for everybody!) Therefore, the system has no solution, and the solution set is $\varnothing$.

PART G: SYSTEMS OF LINEAR EQUATIONS IN THREE UNKNOWNS (SAY *x*, *y*, and *z*)

In Section 8.1, we will solve the following system using matrices:

$$\begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = -28 \\ -x + y = 14 \end{cases}$$

In Section 7.3, a non-matrix preview of the 8.1 method is given.

We call this a system in three unknowns, even though there is no z term in the third equation.

Instead of lines in the xy-plane, we consider planes in xyz-space.

Again, the only possibilities are: 0, 1, or infinitely many solutions.
Real solutions correspond to intersection points common to **all** the planes.
See the figures on p.522.

Solutions are written as ordered triples of the form (x, y, z).
In general, when there are n unknowns, solutions are written as ordered n-tuples.