SECTION 7.4: PARTIAL FRACTIONS

PART A: INTRO

A, B, C, etc. represent unknown real constants.
Assume that our polynomials have real coefficients.

These Examples deal with rational expressions in x, but the methods here extend to rational expressions in y, t, etc.

Review how to add and subtract rational expressions in Section A.4: pp.A38-A39.

Review Example

$$\frac{3}{x-4} - \frac{2}{x+1} = \frac{3\overbrace{(x+1)}^{\text{Think: "Who's missing?"}} - 2(x-4)}{(x-4)(x+1)}$$

$$= \frac{x+11}{x^2-3x-4}$$

How do we reverse this process? In other words, how do we find that the partial fraction decomposition (PFD) for $\frac{x+11}{x^2-3x-4}$ is $\underbrace{\frac{3}{x-4}}_{\text{A partial fraction (PF)}} - \frac{2}{x+1}$?

The PFD Form that we need depends on the **factored form of the denominator**. Here, the denominator is x^2-3x-4, which factors as $(x-4)(x+1)$.

PART B: THE BIG PICTURE

You may want to come back to this Part and Part C after you read the Examples starting on Notes 7.22.

In Section 2.5, we discussed:

"Factoring Over **R**" Theorem

Let $f(x)$ be a nonconstant polynomial in standard form with real coefficients. A complete factorization of $f(x)$ over **R** consists of:

1) Linear factors,
2) Quadratic factors that are **R**-irreducible (see Note below), or
3) Some product of the above, possibly including repeated factors, and
4) Maybe a nonzero constant factor.

Note: A quadratic factor is **R**-irreducible $\Leftrightarrow$
It has no real zeros (or "roots"), and it cannot be nontrivially factored and broken down further over **R** (i.e., using only real coefficients).

Knowing **how** to factor such an $f(x)$ may pose a problem, however! Finding real zeros of $f(x)$ can help you factor $f(x)$; remember the Factor Theorem from Sections 2.2 and 2.3: Notes 2.19 and 2.33.

Example

The hideous polynomial

$$6x^{14}+33x^{13}+45x^{12}+117x^{11}-213x^{10}-2076x^{9}-3180x^{8}-15{,}024x^{7}$$
$$-11{,}952x^{6}-32{,}832x^{5}-18{,}240x^{4}-19{,}968x^{3}-9216x^{2}$$

can factor over **R** as follows:

$$3x^2(x-3)(2x+1)(x+4)^2(x^2+1)(x^2+4)^3$$

Technical Note: There are other factorizations over **R** involving manipulations (like "trading") of constant factors, but we like the fact that the one provided is a factorization over **Z** (the integers), and we have no factors like $(6x+3)$ for which nontrivial GCFs (greatest common factors) can be pulled out.

Let's categorize factors in this factorization:

- 3 is a constant factor.
- We will discuss x^2 last.
- $(x-3)$ is a distinct linear factor.

  It is distinct ("different") in the sense that there are no other $(x-3)$ factors, nor are there constant multiples such as $(2x-6)$.

- $(2x+1)$ is a distinct linear factor.
- $(x+4)^2$ is a [nice] power of a linear factor. Because it can be rewritten as $(x+4)(x+4)$, it is an example of repeated linear factors.
- (x^2+1) is a distinct **R**-irreducible quadratic factor. It is irreducible over **R**, because it has no real zeros (or "roots"); it cannot be nontrivially factored and broken down further over **R**.
- $(x^2+4)^3$ is a [nice] power of an **R**-irreducible quadratic factor. Because it can be rewritten as $(x^2+4)(x^2+4)(x^2+4)$, it is an example of repeated **R**-irreducible quadratic factors.

Warning!

- x^2 actually represents repeated linear factors, because it can be rewritten as $x \cdot x$. You may want to think of it as $(x-0)^2$. We do **not** consider x^2 to be an **R**-irreducible quadratic, because it has a real zero (or root), namely 0.

Why do we care as far as PFDs are concerned?

In Notes 7.14, we showed that $\frac{x+11}{x^2-3x-4}$, or $\frac{x+11}{(x-4)(x+1)}$, can be decomposed as $\frac{3}{x-4}-\frac{2}{x+1}$. We could factor the denominator of the original expression as the product of two distinct linear factors, so we were able to decompose the expression into a sum of two rational expressions with constant numerators and linear denominators.

Note: By "sum," we really mean "sum or difference." Remember that a difference may be reinterpreted as a sum. For example, $7-4=7+(-4)$.

Every proper rational expression of the form $\frac{N(x)}{\text{nonconstant } D(x)}$,
where both $N(x)$ and $D(x)$ are polynomials in x with real coefficients,
has a PFD consisting of a sum of rational expressions ("partial fractions") whose …

… numerators can be constant or linear, and whose …

… denominators can be linear, **R**-irreducible quadratics, or powers thereof.

Note: The PFD for, say, $\frac{1}{x}$ is simply $\frac{1}{x}$. We don't really have a "sum" or a "decomposition" here.

If we have a rational expression that is improper (i.e., the degree of $N(x)$ is not less than the degree of $D(x)$), then Long Division or some other algebraic work is required to express it as either:

- a polynomial, or
- the sum of a polynomial and a proper rational expression. Think:

$$(\textit{polynomial}) \quad + \quad (\textit{proper rational}).$$

We then try to find a PFD for the proper rational expression.

See Notes 7.30-7.32 for an Example.

In Calculus: These PFDs are used when it is preferable (and permissible!) to apply operations (such as integration) term-by-term to a collection of "easy" fractions as opposed to a large, unwieldy fraction. The PFD Method for integration (which is the reverse of differentiation, the process of finding a derivative) is a key topic of Chapter 9 in the Calculus II: Math 151 textbook at Mesa. As it turns out, the comments above imply that we can integrate **any** rational expression up to our ability to factor polynomial denominators. This is a **very** powerful statement!

PART C: PFD FORMS

Let $r(x)$ be a proper rational expression of the form $\dfrac{N(x)}{\text{nonconstant } D(x)}$,

where both $N(x)$ and $D(x)$ are polynomials in x with real coefficients.

Consider a complete factorization of $D(x)$ over **R**. (See Part B.)
Each **linear** or **R-irreducible quadratic** factor of $D(x)$ contributes a term (a partial fraction) to the PFD Form.

Let $m, a, b, c \in \mathbf{R}$.

Category 1a: Distinct Linear Factors; form $(mx+b)$

$(mx+b)$ contributes a term of the form:

$$\frac{A}{mx+b} \quad (A \in \mathbf{R})$$

Note: We may use letters other than A.

Category 1b: Repeated (or Powers of) Linear Factors; form $(mx+b)^n$, $n \in \mathbf{Z}^+$

$(mx+b)^n$ contributes a sum of n terms:

$$\frac{A_1}{mx+b} + \frac{A_2}{(mx+b)^2} + \ldots + \frac{A_n}{(mx+b)^n} \quad (\text{each } A_i \in \mathbf{R})$$

Warning / Think: "Run up to the power." Also observe that each term gets a **numerator** of **constant form**.

Technical Note: $(x+2)$ and $(3x+6)$ do **not** count as distinct linear factors, because they are only separated by a constant factor (3), which can be factored out of the latter.

Note: You can think of Category 1a as a special case of this where $n=1$.

Category 2a: Distinct **R**-Irreducible Quadratic Factors; form $\left(ax^2+bx+c\right)$

$\left(ax^2+bx+c\right)$ contributes a term of the form:

$$\frac{Ax+B}{ax^2+bx+c} \quad \left(A, B \in \mathbf{R}\right)$$

Category 2b: Repeated (or Powers of) **R**-Irreducible Quadratic Factors; form $\left(ax^2+bx+c\right)^n, \; n \in \mathbf{Z}^+$

$\left(ax^2+bx+c\right)^n$ contributes a sum of n terms:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2} + \ldots + \frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n} \quad \left(\text{each } A_i, B_i \in \mathbf{R}\right)$$

Warning / Think: "Run up to the power." Also observe that each term gets a **numerator** of **linear form**, though the numerator may turn out to be just a constant.

Note: You can think of Category 2a as a special case of this where $n=1$.

Example

Find the PFD Form for $\dfrac{1}{x^2(x-4)^2(x^2+1)}$.

You do not have to solve for the unknowns.

Solution

$$\frac{1}{x^2(x-4)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4} + \frac{D}{(x-4)^2} + \frac{Ex+F}{x^2+1}$$

$$(A, B, C, D, E, F \in \mathbf{R})$$

Warning: Remember that x^2 represents repeated linear factors. See Notes 7.16.

We "run up to the power" for both the x^2 and $(x-4)^2$ factors in the denominator.

Because x^2+1 is an **R**-irreducible quadratic, we have a linear form, $Ex+F$, in the corresponding numerator.

Technical Note: If a constant aside from ± 1 can be factored out of the denominator, you can do so immediately. A factor of 3 in the denominator, for example, can be factored out of the overall fraction as a $\frac{1}{3}$. This may help, because you do not want to consider $(x+2)$ and $(3x+6)$, for example, as distinct linear factors. We prefer **complete** factorizations over **R**.

PART D: STEPS; DISTINCT LINEAR FACTORS

Example

Find the PFD for $\frac{x+11}{x^2-3x-4}$. (Let's reverse the work from Part A.)

Solution

Step 1: If the expression is improper, use Long Division to obtain the form: (polynomial) + (proper rational expression).

Technical Note: Synthetic Division works when the denominator is of the form $x-k,\ k \in \mathbf{R}$. In that case, you wouldn't need a PFD!

$\frac{x+11}{x^2-3x-4}$ is proper, so Long Division is unnecessary.

Step 2: Factor the denominator (completely) over **R**.

$$\frac{x+11}{x^2-3x-4} = \frac{x+11}{(x-4)(x+1)}$$

Step 3: Determine the required PFD Form.

The denominator consists of distinct linear factors, so the PFD form is given by:

$$\frac{x+11}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$

Note: The form $\frac{A}{x+1} + \frac{B}{x-4}$ may also be used. The roles of A and B will then be switched in the following work.

Step 4: Multiply both sides of the equation by the LCD (least or lowest common denominator), the denominator on the left.

These steps can be skipped:

$$\cancel{(x-4)}\cancel{(x+1)}\left[\frac{x+11}{\cancel{(x-4)}\cancel{(x+1)}}\right]=\cancel{(x-4)}(x+1)\left[\frac{A}{\cancel{x-4}}\right]+(x-4)\cancel{(x+1)}\left[\frac{B}{\cancel{x+1}}\right]$$

Instead, we can use the "Who's missing?" trick for each term on the right side:

$$x+11=A(x+1)+B(x-4)$$

This is called the basic equation.

Step 5: Solve the basic equation for the unknowns, A and B.

Technical Note: A and B are unique, given the PFD Form. The PFD will be unique up to a reordering of the terms and manipulations of constant factors.

Method 1 ("Plug In"): Plug convenient values for x into the basic equation.

For the correct values of A and B, the basic equation holds true for **all** real values of x, even those values excluded from the domain of the original expression. This can be proven in Calculus.

We would like to choose values for x that will make the "coefficient" of A or B equal to 0.

Plug in $x=-1$:

$$\begin{aligned} x+11 &= A(x+1)+B(x-4) \\ -1+11 &= \cancel{A(-1+1)}^{0}+B(-1-4) \\ 10 &= -5B \\ \boldsymbol{B} &= \boldsymbol{-2} \end{aligned}$$

Plug in $x = 4$:

$$x + 11 = A(x+1) + B(x-4)$$
$$4 + 11 = A(4+1) + \cancel{B(4-4)}^{\,0}$$
$$15 = 5A$$
$$\mathbf{A = 3}$$

Note: Other values for x may be chosen, but you run the risk of having to solve a more complicated system of linear equations.

Method 2 ("Match Coefficients"): Write the right-hand side of the basic equation in standard form, and match (i.e., equate) corresponding coefficients.

$$x + 11 = A(x+1) + B(x-4)$$
$$x + 11 = Ax + A + Bx - 4B$$
$$(1)x + (11) = (A+B)x + (A-4B)$$

The "(1)" coefficient and the parentheses on the left side are optional, but they may help you clearly identify coefficients.

Given that both sides are written in standard form,
the left-hand side is equivalent to the right-hand side $\Leftrightarrow$
every corresponding pair of coefficients of like terms are equal.

We must solve the system:

$$\begin{cases} A + B = 1 \\ A - 4B = 11 \end{cases}$$

See Sections 7.1 and 7.2 for a review of how to solve systems of two linear equations in two unknowns.

We could solve the first equation for A and use the Substitution Method:

$$\begin{cases} A + \ \ B = \ 1 & \Rightarrow \quad A = 1 - B \quad \searrow \\ A - 4B = 11 & \qquad\qquad\qquad\qquad \Rightarrow \quad (1 - B) - 4B = 11 \end{cases}$$

$$1 - 5B = 11$$

$$-10 = 5B$$

$$\mathbf{B = -2}$$

Then,

$$A = 1 - B$$

$$A = 1 - (-2)$$

$$\mathbf{A = 3}$$

Alternately, we could multiply both sides of the first equation by -1 (so that we have opposite coefficients for one of the unknowns) and use the Addition / Elimination Method, in which we "add equations" (really, add equals to equals) to obtain a new equation:

$$\begin{cases} A + \ \ B = \ 1 & \leftarrow \cdot(-1) \\ A - 4B = 11 \end{cases}$$

$$\begin{cases} -A - \ \ B = -1 \\ \ \ A - 4B = \ 11 \end{cases}$$

$$-5B = 10$$

$$\mathbf{B = -2}$$

Now, let's use the original first equation to find A:

$$A + B = 1$$

$$A + (-2) = 1$$

$$\mathbf{A = 3}$$

Note: We may want to combine Methods 1 and 2 ("Plug In" and "Match Coefficients") when solving more complicated problems. Method 1 is usually easier to use, so we often use it first to find as many of the unknowns as we can with ease. (Remember that **any** real value for *x* may be plugged into the basic equation.) We can then plug values of unknowns we have found into the basic equation and use Method 2 to find the values of the remaining unknowns. Method 2 tends to be more directly useful as we discuss more complicated cases.

Step 6: Write out the PFD.

$$\frac{x+11}{x^2-3x-4} \text{ or } \frac{x+11}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$

$$= \frac{3}{x-4} + \frac{-2}{x+1}$$

$$= \mathbf{\frac{3}{x-4} - \frac{2}{x+1}}$$