PART E: REPEATED (OR POWERS OF) LINEAR FACTORS

Example

Find the PFD for $\frac{x^2}{(x+2)^3}$.

Solution

Step 1: The expression is proper, because 2, the degree of $N(x)$, is less than 3, the degree of $D(x)$.

Warning: Imagine that $N(x)$ and $D(x)$ have been written out in standard form before determining the degrees.

Step 2: Factor the denominator over **R**. (Done!)

Step 3: Determine the required PFD Form.

The denominator consists of repeated linear factors, so the PFD Form is given by:

$$\frac{x^2}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

We must "run up to the power."

Step 4: Multiply both sides of the equation by the LCD, $(x+2)^3$, to obtain the basic equation.

$$x^2 = A(x+2)^2 + B(x+2) + C$$

Step 5: Solve the basic equation for the unknowns, A, B, and C.

Plug in $x = -2$:

$$x^2 = A(x+2)^2 + B(x+2) + C$$

$$(-2)^2 = \cancel{A(-2+2)^2}^{\,0} + \cancel{B(-2+2)}^{\,0} + C$$

$$\boldsymbol{C = 4}$$

Updated basic equation; we now know $C = 4$:

$$x^2 = A(x+2)^2 + B(x+2) + 4$$

We can use the "Match Coefficients" Method, or we can plug in a couple of other real values for x, as follows:

Plug in $x = 0$, say:

$$x^2 = A(x+2)^2 + B(x+2) + 4$$

$$(0)^2 = A(0+2)^2 + B(0+2) + 4$$

$$0 = 4A + 2B + 4$$

Let's divide both sides by 2 and switch sides.

$$2A + B + 2 = 0$$

$$2A + B = -2$$

Plug in $x = -1$, say:

$$x^2 = A(x+2)^2 + B(x+2) + 4$$

$$(-1)^2 = A(-1+2)^2 + B(-1+2) + 4$$

$$1 = A + B + 4$$

$$-3 = A + B$$

$$A + B = -3$$

We must solve the system:

$$\begin{cases} 2A + B = -2 \\ A + B = -3 \end{cases}$$

After some work, we find that $\mathbf{A = 1}$ and $\mathbf{B = -4}$.

Step 6: Write out the PFD.

$$\begin{aligned} \frac{x^2}{(x+2)^3} &= \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \\ &= \frac{1}{x+2} + \frac{-4}{(x+2)^2} + \frac{4}{(x+2)^3} \\ &= \mathbf{\frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{4}{(x+2)^3}} \end{aligned}$$

PART F: DISTINCT R-IRREDUCIBLE QUADRATIC FACTORS

Example

$\frac{1}{x^2+1}$ can't be decomposed further using a PFD over **R**, because the denominator is, itself, an irreducible quadratic over **R** (i.e., it has no real roots).

Technical Note: There is such a thing as a PFD over **C**.

Example

Find the PFD for $\frac{2x^4 - 2x^3 + 10x^2 - 3x + 9}{2x^3 + 3x}$.

Solution

Step 1: The expression is improper, because 4, the degree of $N(x)$, is not less than 3, the degree of $D(x)$.

After performing the Long Division (left up to you!), we obtain:

$$\frac{2x^4 - 2x^3 + 10x^2 - 3x + 9}{2x^3 + 3x} = x - 1 + \frac{7x^2 + 9}{2x^3 + 3x}$$

Step 2: Factor the denominator over **R**.

Warning: We will ignore the polynomial part, $x - 1$, for now, but don't forget about it when giving your final answer.

$$\frac{7x^2 + 9}{2x^3 + 3x} = \frac{7x^2 + 9}{x(2x^2 + 3)}$$

<u>Step 3</u>: Determine the required PFD Form.

$$\frac{7x^2+9}{x\left(2x^2+3\right)} = \frac{A}{x} + \frac{Bx+C}{2x^2+3}; \text{ remember } x-1$$

You should remind yourself of the $x-1$ polynomial part here, because you will refer to this step when you write out the PFD at the end.

<u>Step 4</u>: Multiply both sides of the equation by the LCD, $x\left(2x^2+3\right)$, to obtain the basic equation.

$$7x^2+9 = A\left(2x^2+3\right) + \left(Bx+C\right)x$$

<u>Step 5</u>: Solve the basic equation for the unknowns, A, B, and C.

Plug in $x=0$:

$$7x^2+9 = A\left(2x^2+3\right) + \left(Bx+C\right)x$$

$$7(0)^2+9 = A\left[2(0)^2+3\right] + \cancel{\left[B(0)+C\right](0)}^{0}$$

$$9 = 3A$$

$$\boldsymbol{A=3}$$

Updated basic equation; we now know $A=3$:

$$7x^2+9 = 3\left(2x^2+3\right) + \left(Bx+C\right)x$$

Let's use the "Match Coefficients" Method.

$$7x^2+0x+9 = 6x^2+9+Bx^2+Cx$$

$$(7)x^2+(0)x+(9) = (6+B)x^2+(C)x+(9)$$

<u>Tip</u>: Inserting the $+0x$ on the left side may be helpful.

<u>Note</u>: The "9"s better match up, or else we're in trouble!

Match the x^2 coefficients:

$$7 = 6 + B$$
$$\boldsymbol{B = 1}$$

Match the x coefficients:

$$0 = C$$
$$\boldsymbol{C = 0}$$

<u>Step 6</u>: Write out the PFD.

<u>Warning</u>: Don't forget the polynomial part, $x - 1$. That's why we had that reminder back in Step 3.

$$\begin{aligned} \frac{2x^4 - 2x^3 + 10x^2 - 3x + 9}{2x^3 + 3x} &= x - 1 \ + \ \frac{A}{x} \ + \ \frac{Bx + C}{2x^2 + 3} \\ &= x - 1 \ + \ \frac{3}{x} \ + \ \frac{1x + 0}{2x^2 + 3} \\ &= \boldsymbol{x - 1 \ + \ \frac{3}{x} \ + \ \frac{x}{2x^2 + 3}} \end{aligned}$$

PART G: REPEATED (OR POWERS OF) R-IRREDUCIBLE QUADRATIC FACTORS

Example

Find the PFD for $\dfrac{2x^3 - x^2 + 2x + 2}{\left(x^2+1\right)^2}$.

Solution

Step 1: The expression is proper, because 3, the degree of $N(x)$, is less than 4, the degree of $D(x)$.

Warning: Imagine that $N(x)$ and $D(x)$ have been written out in standard form before determining their degrees.

Step 2: Factor the denominator over **R**. (Done!)

Observe that $\left(x^2+1\right)$ has no real zeros.

Step 3: Determine the required PFD Form.

$$\frac{2x^3 - x^2 + 2x + 2}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

We must "run up to the power."

Step 4: Multiply both sides of the equation by the LCD, $\left(x^2+1\right)^2$, to obtain the basic equation.

$$2x^3 - x^2 + 2x + 2 = (Ax+B)\left(x^2+1\right) + (Cx+D)$$

<u>Step 5</u>: Solve the basic equation for the unknowns, A, B, C, and D.

Let's use the "Match Coefficients" Method immediately.

$$2x^3 - 1x^2 + 2x + 2 \;=\; Ax^3 + Ax + Bx^2 + B + Cx + D$$
$$(2)x^3 + (-1)x^2 + (2)x + (2) = (A)x^3 + (B)x^2 + (A+C)x + (B+D)$$

<u>Note</u>: If you use the optional parentheses on the left side, remember to separate your terms with "+" signs in order to avoid confusion.

<u>Note</u>: You may want to mark off terms as you collect like terms on the right side.

By matching the x^3 coefficients and the x^2 coefficients, we immediately obtain:

$$\boldsymbol{A = 2}$$
$$\boldsymbol{B = -1}$$

Match the x coefficients, and use the previous info:

$$A + C = 2$$
$$2 + C = 2$$
$$\boldsymbol{C = 0}$$

Match the constant terms, and use the previous info:

$$B + D = 2$$
$$-1 + D = 2$$
$$\boldsymbol{D = 3}$$

We have solved the system:

$$\begin{cases} A = 2 \\ B = -1 \\ A + C = 2 \\ B + D = 2 \end{cases}$$

Step 6: Write out the PFD.

$$\frac{2x^3 - x^2 + 2x + 2}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$= \frac{2x-1}{x^2+1} + \frac{0x+3}{\left(x^2+1\right)^2}$$

$$= \mathbf{\frac{2x-1}{x^2+1} + \frac{3}{\left(x^2+1\right)^2}}$$

Historical Note on p.536: John Bernoulli introduced the methods of this section.