

PART E: REPEATED (OR POWERS OF) LINEAR FACTORSExample

Find the PFD for $\frac{x^2}{(x+2)^3}$.

Solution

Step 1: The expression is proper, because 2, the degree of $N(x)$, is less than 3, the degree of $D(x)$.

Warning: Imagine that $N(x)$ and $D(x)$ have been written out in standard form before determining the degrees.

Step 2: Factor the denominator over **R**. (Done!)

Step 3: Determine the required PFD Form.

The denominator consists of repeated linear factors, so the PFD Form is given by:

$$\frac{x^2}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

We must “run up to the power.”

Step 4: Multiply both sides of the equation by the LCD, $(x+2)^3$, to obtain the basic equation.

$$x^2 = A(x+2)^2 + B(x+2) + C$$

Step 5: Solve the basic equation for the unknowns, A , B , and C .

Plug in $x = -2$:

$$x^2 = A(x+2)^2 + B(x+2) + C$$

$$(-2)^2 = A(\cancel{-2+2})^2 + B(\cancel{-2+2}) + C$$

$$C = 4$$

Updated basic equation; we now know $C = 4$:

$$x^2 = A(x+2)^2 + B(x+2) + 4$$

We can use the “Match Coefficients” Method, or we can plug in a couple of other real values for x , as follows:

Plug in $x = 0$, say:

$$x^2 = A(x+2)^2 + B(x+2) + 4$$

$$(0)^2 = A(0+2)^2 + B(0+2) + 4$$

$$0 = 4A + 2B + 4$$

Let's divide both sides by 2
and switch sides.

$$2A + B + 2 = 0$$

$$2A + B = -2$$

Plug in $x = -1$, say:

$$x^2 = A(x+2)^2 + B(x+2) + 4$$

$$(-1)^2 = A(-1+2)^2 + B(-1+2) + 4$$

$$1 = A + B + 4$$

$$-3 = A + B$$

$$A + B = -3$$

We must solve the system:

$$\begin{cases} 2A + B = -2 \\ A + B = -3 \end{cases}$$

After some work, we find that $A = 1$ and $B = -4$.

Step 6: Write out the PFD.

$$\begin{aligned} \frac{x^2}{(x+2)^3} &= \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \\ &= \frac{1}{x+2} + \frac{-4}{(x+2)^2} + \frac{4}{(x+2)^3} \\ &= \frac{\mathbf{1}}{x+2} - \frac{\mathbf{4}}{(x+2)^2} + \frac{\mathbf{4}}{(x+2)^3} \end{aligned}$$

PART F: DISTINCT R-IRREDUCIBLE QUADRATIC FACTORSExample

$\frac{1}{x^2 + 1}$ can't be decomposed further using a PFD over \mathbf{R} , because the denominator is, itself, an irreducible quadratic over \mathbf{R} (i.e., it has no real roots).

Technical Note: There is such a thing as a PFD over \mathbf{C} .

Example

Find the PFD for $\frac{2x^4 - 2x^3 + 10x^2 - 3x + 9}{2x^3 + 3x}$.

Solution

Step 1: The expression is improper, because 4, the degree of $N(x)$, is not less than 3, the degree of $D(x)$.

After performing the Long Division (left up to you!), we obtain:

$$\frac{2x^4 - 2x^3 + 10x^2 - 3x + 9}{2x^3 + 3x} = x - 1 + \frac{7x^2 + 9}{2x^3 + 3x}$$

Step 2: Factor the denominator over \mathbf{R} .

Warning: We will ignore the polynomial part, $x - 1$, for now, but don't forget about it when giving your final answer.

$$\frac{7x^2 + 9}{2x^3 + 3x} = \frac{7x^2 + 9}{x(2x^2 + 3)}$$

Step 3: Determine the required PFD Form.

$$\frac{7x^2 + 9}{x(2x^2 + 3)} = \frac{A}{x} + \frac{Bx + C}{2x^2 + 3}; \text{ remember } x - 1$$

You should remind yourself of the $x - 1$ polynomial part here, because you will refer to this step when you write out the PFD at the end.

Step 4: Multiply both sides of the equation by the LCD, $x(2x^2 + 3)$, to obtain the basic equation.

$$7x^2 + 9 = A(2x^2 + 3) + (Bx + C)x$$

Step 5: Solve the basic equation for the unknowns, A , B , and C .

Plug in $x = 0$:

$$\begin{aligned} 7x^2 + 9 &= A(2x^2 + 3) + (Bx + C)x \\ 7(0)^2 + 9 &= A[2(0)^2 + 3] + \cancel{[B(0) + C]}(0)^0 \\ 9 &= 3A \\ A &= 3 \end{aligned}$$

Updated basic equation; we now know $A = 3$:

$$7x^2 + 9 = 3(2x^2 + 3) + (Bx + C)x$$

Let's use the "Match Coefficients" Method.

$$\begin{aligned} 7x^2 + 0x + 9 &= 6x^2 + 9 + Bx^2 + Cx \\ (7)x^2 + (0)x + (9) &= (6 + B)x^2 + (C)x + (9) \end{aligned}$$

Tip: Inserting the $+ 0x$ on the left side may be helpful.

Note: The "9"s better match up, or else we're in trouble!

Match the x^2 coefficients:

$$7 = 6 + B$$

$$\mathbf{B = 1}$$

Match the x coefficients:

$$0 = C$$

$$\mathbf{C = 0}$$

Step 6: Write out the PFD.

Warning: Don't forget the polynomial part, $x - 1$. That's why we had that reminder back in Step 3.

$$\begin{aligned}\frac{2x^4 - 2x^3 + 10x^2 - 3x + 9}{2x^3 + 3x} &= x - 1 + \frac{A}{x} + \frac{Bx + C}{2x^2 + 3} \\ &= x - 1 + \frac{3}{x} + \frac{1x + 0}{2x^2 + 3} \\ &= \mathbf{x - 1} + \frac{\mathbf{3}}{\mathbf{x}} + \frac{\mathbf{x}}{\mathbf{2x^2 + 3}}\end{aligned}$$

PART G: REPEATED (OR POWERS OF) R-IRREDUCIBLE QUADRATIC FACTORSExample

Find the PFD for $\frac{2x^3 - x^2 + 2x + 2}{(x^2 + 1)^2}$.

Solution

Step 1: The expression is proper, because 3, the degree of $N(x)$, is less than 4, the degree of $D(x)$.

Warning: Imagine that $N(x)$ and $D(x)$ have been written out in standard form before determining their degrees.

Step 2: Factor the denominator over **R**. (Done!)

Observe that $(x^2 + 1)$ has no real zeros.

Step 3: Determine the required PFD Form.

$$\frac{2x^3 - x^2 + 2x + 2}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

We must “run up to the power.”

Step 4: Multiply both sides of the equation by the LCD, $(x^2 + 1)^2$, to obtain the basic equation.

$$2x^3 - x^2 + 2x + 2 = (Ax + B)(x^2 + 1) + (Cx + D)$$

Step 5: Solve the basic equation for the unknowns, A , B , C , and D .

Let's use the "Match Coefficients" Method immediately.

$$2x^3 - 1x^2 + 2x + 2 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

$$(2)x^3 + (-1)x^2 + (2)x + (2) = (A)x^3 + (B)x^2 + (A + C)x + (B + D)$$

Note: If you use the optional parentheses on the left side, remember to separate your terms with "+" signs in order to avoid confusion.

Note: You may want to mark off terms as you collect like terms on the right side.

By matching the x^3 coefficients and the x^2 coefficients, we immediately obtain:

$$A = 2$$

$$B = -1$$

Match the x coefficients, and use the previous info:

$$A + C = 2$$

$$2 + C = 2$$

$$C = 0$$

Match the constant terms, and use the previous info:

$$B + D = 2$$

$$-1 + D = 2$$

$$D = 3$$

We have solved the system:

$$\begin{cases} A = 2 \\ B = -1 \\ A + C = 2 \\ B + D = 2 \end{cases}$$

Step 6: Write out the PFD.

$$\begin{aligned}\frac{2x^3 - x^2 + 2x + 2}{(x^2 + 1)^2} &= \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} \\ &= \frac{2x - 1}{x^2 + 1} + \frac{0x + 3}{(x^2 + 1)^2} \\ &= \frac{2x - 1}{x^2 + 1} + \frac{3}{(x^2 + 1)^2}\end{aligned}$$

Historical Note on p.536: John Bernoulli introduced the methods of this section.