CHAPTER 8: MATRICES and DETERMINANTS

The material in this chapter will be covered in your Linear Algebra class (Math 254 at Mesa).

SECTION 8.1: MATRICES and SYSTEMS OF EQUATIONS

PART A: MATRICES

A matrix is basically an organized box (or "array") of numbers (or other expressions). In this chapter, we will typically assume that our matrices contain only numbers.

Example

Here is a matrix of size 2×3 ("2 by 3"), because it has 2 rows and 3 columns:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

The matrix consists of 6 entries or elements.

In general, an $m \times n$ matrix has m rows and n columns and has mn entries.

Example

Here is a matrix of size 2×2 (an order 2 square matrix):

$$\begin{bmatrix} \mathbf{4} & -1 \\ 3 & \mathbf{2} \end{bmatrix}$$

The **boldfaced** entries lie on the main diagonal of the matrix.
(The other diagonal is the skew diagonal.)

PART B: THE AUGMENTED MATRIX FOR A SYSTEM OF LINEAR EQUATIONS

Example

Write the augmented matrix for the system: $\begin{cases} 3x + 2y + z = 0 \\ -2x - z = 3 \end{cases}$

Solution

Preliminaries:

Make sure that the equations are in (what we refer to now as) standard form, meaning that …

• All of the variable terms are on the left side (with x, y, and z ordered alphabetically), and

• There is only one constant term, and it is on the right side.

Line up like terms vertically.

Here, we will rewrite the system as follows:

$$\begin{cases} 3x + 2y + z = 0 \\ -2x - z = 3 \end{cases}$$

(Optional) Insert "1"s and "0"s to clarify coefficients.

$$\begin{cases} 3x + 2y + 1z = 0 \\ -2x + 0y - 1z = 3 \end{cases}$$

Warning: Although this step is not necessary, people often mistake the coefficients on the z terms for "0"s.

Write the augmented matrix:

$$\begin{array}{c} \text{Coefficients of} \\ \begin{array}{ccc} x & y & z \end{array} \end{array} \quad \begin{array}{c} \text{Right} \\ \text{sides} \end{array}$$

$$\underbrace{\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ -2 & 0 & -1 & 3 \end{array}\right]}_{\text{Augmented matrix}}$$

<u>Coefficient matrix</u> <u>Right-hand side (RHS)</u>

<u>Augmented matrix</u>

We may refer to the first three columns as the *x*-column, the *y*-column, and the *z*-column of the coefficient matrix.

<u>Warning</u>: If you do not insert "1"s and "0"s, you may want to read the equations and fill out the matrix row by row in order to minimize the chance of errors. Otherwise, it may be faster to fill it out column by column.

The augmented matrix is an efficient representation of a system of linear equations, although the names of the variables are hidden.

PART C: ELEMENTARY ROW OPERATIONS (EROs)

Recall from Algebra I that equivalent equations have the same solution set.

Example

Solve: $2x-1=5$

$$\begin{aligned} 2x-1&=5 \\ 2x&=6 \\ x&=3 \quad \Rightarrow \quad \text{Solution set is } \{3\}. \end{aligned}$$

To solve the first equation, we write a sequence of equivalent equations until we arrive at an equation whose solution set is obvious.

The steps of adding 1 to both sides of the first equation and of dividing both sides of the second equation by 2 are like “legal chess moves” that allowed us to maintain equivalence (i.e., to preserve the solution set).

Similarly, equivalent systems have the same solution set.

Elementary Row Operations (EROs) represent the legal moves that allow us to write a sequence of row-equivalent matrices (corresponding to equivalent systems) until we obtain one whose corresponding solution set is easy to find. There are three types of EROs:

1) **Row Reordering**

Example

Consider the system: $\begin{cases} 3x - y = 1 \\ x + y = 4 \end{cases}$

If we switch (i.e., interchange) the two equations, then the solution set is not disturbed:

$$\begin{cases} x + y = 4 \\ 3x - y = 1 \end{cases}$$

This suggests that, when we solve a system using augmented matrices, …

We can switch any two rows.

Before:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 3 & -1 & 1 \\ 1 & 1 & 4 \end{array} \right]$$

Here, we switch rows R_1 and R_2, which we denote by: $R_1 \leftrightarrow R_2$

After:

$$\begin{matrix} \text{new } R_1 \\ \text{new } R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & -1 & 1 \end{array} \right]$$

In general, we can reorder the rows of an augmented matrix in any order.

Warning: Do **not** reorder columns; in the coefficient matrix, that will change the order of the corresponding variables.

2) <u>Row Rescaling</u>

<u>Example</u>

Consider the system: $\begin{cases} \frac{1}{2}x + \frac{1}{2}y = 3 \\ \qquad\quad\ \ y = 4 \end{cases}$

If we multiply "through" both sides of the first equation by 2, then we obtain an equivalent equation and, overall, an equivalent system:

$$\begin{cases} x + y = 6 \\ \quad\ \ y = 4 \end{cases}$$

This suggests that, when we solve a system using augmented matrices, …

> **We can multiply (or divide) "through" a row by any nonzero constant.**

Before:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1/2 & 1/2 & 3 \\ 0 & 1 & 4 \end{array} \right]$$

Here, we multiply through R_1 by 2, which we denote by: $R_1 \leftarrow 2 \cdot R_1$, or $\left(\text{new } R_1\right) \leftarrow 2 \cdot \left(\text{old } R_1\right)$

After:

$$\begin{matrix} \text{new } R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 4 \end{array} \right]$$

3) Row Replacement

(This is perhaps poorly named, since ERO types 1 and 2 may also be viewed as "row replacements" in a literal sense.)

When we solve a system using augmented matrices, …

We can add a multiple of one row to another row.

Technical Note: This combines ideas from the Row Rescaling ERO and the Addition Method from Chapter 7.

Example

Consider the system:
$$\begin{cases} x + 3y = 3 \\ -2x + 5y = 16 \end{cases}$$

Before:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 3 & 3 \\ \mathbf{-2} & 5 & 16 \end{array} \right]$$

Note: We will sometimes boldface items for purposes of clarity.

It turns out that we want to add twice the first row to the second row, because we want to replace the "$\mathbf{-2}$" with a "0."

We denote this by:

$$R_2 \leftarrow R_2 + 2 \cdot R_1, \text{ or } (\text{new } R_2) \leftarrow (\text{old } R_2) + 2 \cdot R_1$$

old R_2	-2	5		16
$+2 \cdot R_1$	2	6		6
new R_2	**0**	**11**		**22**

Warning: It is **highly** advised that you write out the table! People often rush through this step and make mechanical errors.

Warning: Although we can also **subtract** a multiple of one row from another row, we generally prefer to **add**, instead, even if that means that we multiply "through" a row by a negative number. Errors are common when people subtract.

After:

$$\begin{matrix} \text{old } R_1 \\ \text{new } R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 11 & 22 \end{array}\right]$$

Note: In principle, you could replace the old R_1 with the rescaled version, but it turns out that we like having that "1" in the upper left hand corner!

If matrix B is obtained from matrix A after applying one or more EROs, then we call A and B row-equivalent matrices, and we write $A \sim B$.

Example

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 7 & 8 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 7 & 8 & 9 \\ 1 & 2 & 3 \end{array}\right]$$

Row-equivalent augmented matrices correspond to equivalent systems, assuming that the underlying variables (corresponding to the columns of the coefficient matrix) stay the same and are in the same order.

PART D: GAUSSIAN ELIMINATION (WITH BACK-SUBSTITUTION)

This is a method for solving systems of linear equations.

Historical Note: This method was popularized by the great mathematician Carl Gauss, but the Chinese were using it as early as 200 BC.

Steps

Given a square system (i.e., a system of n linear equations in n unknowns for some $n \in \mathbf{Z}^{+}$; we will consider other cases later) …

1) Write the **augmented matrix**.

2) **Use EROs** to write a sequence of row-equivalent matrices until you get one in the form:

$$\left[\begin{array}{cccc|c} 1 & & ? & & \\ & 1 & & & ? \\ 0 & & \ddots & & \\ & & & 1 & \end{array}\right]$$

If we begin with a square system, then all of the coefficient matrices will be square.

We want "1"s along the main diagonal and "0"s all below.
The other entries are "wild cards" that can potentially be any real numbers.

This is the form that we are aiming for. Think of this as "checkmate" or "the top of the jigsaw puzzle box" or "the TARGET" (like in a trig ID).

Warning: As you perform EROs and this form crystallizes and emerges, you usually want to avoid "undoing" the good work you have already done. For example, if you get a "1" in the upper left corner, you usually want to preserve it. For this reason, it is often a good strategy to **"correct" the columns from left to right** (that is, from the leftmost column to the rightmost column) in the coefficient matrix. Different strategies may work better under different circumstances.

For now, assume that we have succeeded in obtaining this form; this means that the system has exactly one solution.

What if it is impossible for us to obtain this form? We shall discuss this matter later (starting with Notes 8.21).

3) Write the **new system**, complete with variables.

   This system will be equivalent to the given system, meaning that they share the same solution set. The new system should be **easy to solve** if you …

4) Use **back-substitution** to find the values of the unknowns.

   We will discuss this later.

5) Write the solution as an ordered *n*-tuple (pair, triple, etc.).

6) **Check** the solution in the given system. (Optional)

   Warning: This check will not capture other solutions if there are, in fact, infinitely many solutions.

Technical Note: This method actually works with complex numbers in general.

Warning: You may want to quickly check each of your steps before proceeding. A single mistake can have massive consequences that are difficult to correct.

Example

Solve the system: $\begin{cases} 4x - y = 13 \\ x - 2y = 5 \end{cases}$

Solution

Step 1) Write the augmented matrix.

You may first want to insert "1"s and "0"s where appropriate.

$$\begin{cases} 4x - 1y = 13 \\ 1x - 2y = 5 \end{cases}$$

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} \mathbf{4} & -1 & 13 \\ 1 & -2 & 5 \end{array} \right]$$

Note: It's up to you if you want to write the "R_1" and the "R_2."

Step 2) Use EROs until we obtain the desired form: $\left[\begin{array}{cc|c} 1 & ? & ? \\ 0 & 1 & ? \end{array} \right]$

Note: There may be different "good" ways to achieve our goal.

We want a "1" to replace the "**4**" in the upper left.
Dividing through R_1 by 4 will do it, but we will then end up with fractions. Sometimes, we can't avoid fractions. Here, we can.

Instead, let's switch the rows.

$$R_1 \leftrightarrow R_2$$

Warning: You should keep a record of your EROs. This will reduce eyestrain and frustration if you want to check your work!

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ \mathbf{4} & -1 & 13 \end{array} \right]$$

We now want a "0" to replace the "**4**" in the bottom left.
Remember, we generally want to "correct" columns from left to right, so we will attack the position containing the -1 later.

We cannot multiply through a row by 0.

Instead, we will use a row replacement ERO that exploits the "1" in the upper left to "kill off" the "**4**." This really represents the <u>elimination</u> of the *x* term in what is now the second equation in our system.

$$\left(\text{new } R_2\right) \leftarrow \left(\text{old } R_2\right) + \left(-4\right) \cdot R_1$$

The notation above is really unnecessary if you show the work below:

old R_2	4	-1		13
$+\left(-4\right) \cdot R_1$	-4	8		-20
new R_2	**0**	**7**		**−7**

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & \mathbf{7} & -7 \end{array} \right]$$

We want a "1" to replace the "**7**."
We will divide through R_2 by 7, or, equivalently, we will multiply through R_2 by $\frac{1}{7}$:

$$R_2 \leftarrow \frac{1}{7} \cdot R_2 \text{, or}$$

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & \mathbf{7} & -7 \end{array} \right] \begin{matrix} \\ \leftarrow \div 7 \end{matrix}$$

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -1 \end{array} \right]$$

We now have our desired form.

Technical Note: What's best for computation by hand may not be best for computer algorithms that attempt to maximize precision and accuracy. For example, the strategy of partial pivoting would have kept the "4" in the upper left position of the original matrix and would have used it to eliminate the "1" below.

Note: Some books remove the requirement that the entries along the main diagonal all have to be "1"s. However, when we refer to Gaussian Elimination, we will require that they all be "1"s.

Step 3) Write the new system.

You may want to write down the variables on top of their corresponding columns.

$$\begin{array}{cc} x & y \end{array}$$
$$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -1 \end{array}\right]$$

$$\begin{cases} x - 2y = 5 \\ \quad\;\; y = -1 \end{cases} \quad \uparrow$$

This is called an upper triangular system, which is very easy to solve if we …

Step 4) Use back-substitution.

We start at the bottom, where we immediately find that $\boldsymbol{y = -1}$.

We then work our way up the system, plugging in values for unknowns along the way whenever we know them.

$$\begin{aligned} x - 2y &= 5 \\ x - 2(-1) &= 5 \\ x + 2 &= 5 \\ \boldsymbol{x} &= \mathbf{3} \end{aligned}$$

Step 5) Write the solution.

The solution set is: $\{(\mathbf{3}, \mathbf{-1})\}$

Books are often content with omitting the $\{\ \}$ brace symbols. Ask your instructor, though.

Warning: Observe that the order of the coordinates is the **reverse** of the order in which we found them in the back-substitution procedure.

Step 6) Check. (Optional)

Given system: $\begin{cases} 4x - \ \ y = 13 \\ \ \ x - 2y = \ \ 5 \end{cases}$

$$\begin{cases} 4(3) - \ (-1) = 13 \\ \ \ (3) - 2(-1) = \ \ 5 \end{cases}$$

$$\begin{cases} 13 = 13 \\ \ \ 5 = \ \ 5 \end{cases}$$

Our solution checks out.

Example (#62 on p.556)

Solve the system: $\begin{cases} 2x+2y-z= 2 \\ x-3y+z=-28 \\ -x+ y = 14 \end{cases}$

Solution

Step 1) Write the augmented matrix.

You may first want to insert "1"s and "0"s where appropriate.

$$\begin{cases} 2x+2y-1z= 2 \\ 1x-3y+1z=-28 \\ -1x+1y+0z= 14 \end{cases}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \left[\begin{array}{ccc|c} \mathbf{2} & 2 & -1 & 2 \\ 1 & -3 & 1 & -28 \\ -1 & 1 & 0 & 14 \end{array}\right]$$

Step 2) Use EROs until we obtain the desired form: $\left[\begin{array}{ccc|c} 1 & ? & ? & ? \\ 0 & 1 & ? & ? \\ 0 & 0 & 1 & ? \end{array}\right]$

We want a "1" to replace the "**2**" in the upper left corner.
Dividing through R_1 by 2 would do it, but we would then end up with a fraction.

Instead, let's switch the first two rows.

$$R_1 \leftrightarrow R_2$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ \mathbf{2} & 2 & -1 & 2 \\ \mathbf{-1} & 1 & 0 & 14 \end{array}\right]$$

We now want to "eliminate down" the first column by using the "1" in the upper left corner to "kill off" the boldfaced entries and turn them into "0"s.

Warning: Performing more than one ERO before writing down a new matrix often risks mechanical errors. However, when eliminating down a column, we can usually perform several row replacement EROs without confusion before writing a new matrix. (The same is true of multiple row rescalings and of row reorderings, which can represent multiple row interchanges.) Mixing ERO types before writing a new matrix is probably a bad idea, though!

old R_2	2	2	-1		2
$+(-2)\cdot R_1$	-2	6	-2		56
new R_2	**0**	**8**	**-3**		**58**

old R_3	-1	1	0		14
$+R_1$	1	-3	1		-28
new R_3	**0**	**-2**	**1**		**-14**

Now, write down the new matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & 8 & -3 & 58 \\ 0 & -2 & 1 & -14 \end{array}\right]$$

The first column has been "corrected." From a strategic perspective, we may now think of the first row and the first column (in blue) as "locked in." (EROs that change the entries therein are not necessarily "wrong," but you may be in danger of being taken further away from the desired form.)

We will now focus on the second column. We want:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{1} & ? & ? \\ 0 & \mathbf{0} & ? & ? \end{array}\right]$$

Here is our current matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{8} & -3 & 58 \\ 0 & \mathbf{-2} & 1 & -14 \end{array}\right]$$

If we use the "$\mathbf{-2}$" to kill off the "**8**," we can avoid fractions for the time being. Let's first switch R_2 and R_3 so that we don't get confused when we do this. (We're used to eliminating **down** a column.)

Technical Note: The computer-based strategy of partial pivoting would use the "**8**" to kill off the "$\mathbf{-2}$," since the "**8**" is larger in absolute value.

$$R_2 \leftrightarrow R_3$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{-2} & 1 & -14 \\ 0 & \mathbf{8} & -3 & 58 \end{array}\right]$$

Now, we will use a row replacement ERO to eliminate the "**8**."

old R_3	0	8	-3		58
$+4 \cdot R_2$	0	-8	4		-56
new R_3	**0**	**0**	**1**		**2**

Warning: Don't ignore the "0"s on the left; otherwise, you may get confused.

Now, write down the new matrix:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{-2} & 1 & -14 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Once we get a "1" where the "$\mathbf{-2}$" is, we'll have our desired form. We are fortunate that we already have a "1" at the bottom of the third column, so we won't have to "correct" it.

We will divide through R_2 by -2, or, equivalently, we will multiply through R_2 by $-\frac{1}{2}$.

$$R_2 \leftarrow \left(-\frac{1}{2}\right) \cdot R_2, \text{ or}$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & \mathbf{-2} & 1 & -14 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \\ \leftarrow \div(-2) \\ \\ \end{array}$$

We finally obtain a matrix in our desired form:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & 1 & -1/2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Step 3) Write the new system.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & -3 & 1 & -28 \\ 0 & 1 & -1/2 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array}$$

$$\begin{cases} x - 3y \quad + z = -28 \\ \qquad y - \frac{1}{2}z = \quad 7 \quad \uparrow \\ \qquad\qquad\quad z = \quad 2 \quad \uparrow \end{cases}$$

Step 4) Use back-substitution.

We immediately have: $\boldsymbol{z = 2}$

Use $z = 2$ in the second equation:

$$\begin{aligned} y - \frac{1}{2}z &= 7 \\ y - \frac{1}{2}(2) &= 7 \\ y - 1 &= 7 \\ \boldsymbol{y} &\boldsymbol{= 8} \end{aligned}$$

Use $y = 8$ and $z = 2$ in the first equation:

$$\begin{aligned} x - 3y + z &= -28 \\ x - 3(8) + (2) &= -28 \\ x - 24 + 2 &= -28 \\ x - 22 &= -28 \\ \boldsymbol{x} &\boldsymbol{= -6} \end{aligned}$$

Step 5) Write the solution.

The solution set is: $\left\{\left(\mathbf{-6, 8, 2}\right)\right\}$

Warning: Remember that the order of the coordinates is the **reverse** of the order in which we found them in the back-substitution procedure.

Step 6) Check. (Optional)

Given system: $\begin{cases} 2x+2y-z = 2 \\ x-3y+z = -28 \\ -x+y = 14 \end{cases}$

$$\begin{cases} 2(-6)+2(8)-(2) = 2 \\ (-6)-3(8)+(2) = -28 \\ -(-6)+(8) = 14 \end{cases}$$

$$\begin{cases} 2 = 2 \\ -28 = -28 \\ 14 = 14 \end{cases}$$

Our solution checks out.

PART E: WHEN DOES A SYSTEM HAVE NO SOLUTION?

If we **ever** get a row of the form:

$$0 \quad 0 \quad \cdots \quad 0 \quad \big| \quad (\text{non-0 constant}),$$

then STOP! We know at this point that the solution set is $\varnothing$.

Example

Solve the system: $\begin{cases} x + y = 1 \\ x + y = 4 \end{cases}$

Solution

The augmented matrix is:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 4 \end{array} \right]$$

We can quickly subtract R_1 from R_2. We then obtain:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 3 \end{array} \right]$$

The new R_2 implies that the solution set is $\varnothing$.

Comments: This is because R_2 corresponds to the equation $0 = 3$, which cannot hold true for any pair (x, y).

If we get a row of all "0"s, such as:

$$0 \quad 0 \quad \cdots \quad 0 \;\big|\; 0,$$

then what does that imply? The story is more complicated here.

Example

Solve the system: $\begin{cases} x + y = 4 \\ x + y = 4 \end{cases}$

Solution

The augmented matrix is:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & 1 & 4 \end{array} \right]$$

We can quickly subtract R_1 from R_2. We then obtain:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 0 \end{array} \right]$$

The corresponding system is then:

$$\begin{cases} x + y = 4 \\ \quad\;\; 0 = 0 \end{cases}$$

The equation $0 = 0$ is pretty easy to satisfy. All ordered pairs (x, y) satisfy it. In principle, we could delete this equation from the system. However, we tend not to delete rows in an augmented matrix, even if they consist of nothing but "0"s. The idea of changing the size of a matrix creeps us out.

The solution set is:

$$\left\{ (x, y) \,\middle|\, x + y = 4 \right\}$$

The system has infinitely many solutions; they correspond to all of the points on the line $x + y = 4$.

However, a row of all "0"s does **not** automatically imply that the corresponding system has infinitely many solutions.

Example

Consider the augmented matrix:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

Because of R_1, the corresponding system actually has no solution.

See Notes 7.12 for a similar example.

The augmented matrices we have seen in this Part are **not** row equivalent to any matrix of the form

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & ? & ? \\ 0 & 1 & ? \end{array} \right].$$

There was no way to get that desired form using EROs.

What form do we aim for, then?

PART F: ROW-ECHELON FORM FOR A MATRIX

If it is impossible for us to obtain the form

$$\left[\begin{array}{ccc|c} 1 & & ? & \\ & 1 & & ? \\ 0 & & \ddots & \\ & & & 1 \end{array}\right]$$

(maybe because our coefficient matrix isn't even square), then what do we aim for? We aim for row-echelon form; in fact, the above form is a special case of row-echelon form.

Properties of a Matrix in Row-Echelon Form

1) If there are any "all-0" rows, then they must be at the bottom of the matrix.

Aside from these "all-0" rows,

2) Every row must have a "1" (called a "leading 1") as its leftmost non-0 entry.

3) The "leading 1"s must "flow down and to the right."

More precisely: The "leading 1" of a row must be in a column to the right of the "leading 1"s of all higher rows.

Example

The matrix below is in Row-Echelon Form:

$$\left[\begin{array}{ccccc|c} \mathbf{1} & 3 & 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 9 & 2 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The "leading **1**"s are boldfaced.
The "1" in the upper right corner is **not** a "leading 1."

PART G: REDUCED ROW-ECHELON (RRE) FORM FOR A MATRIX

This is a special case of Row-Echelon Form.

Properties of a Matrix in Reduced Row-Echelon (RRE) Form

1-3) It is in Row-Echelon form. (See Part F.)

4) Each "leading 1" has all "0"s elsewhere in its column.

Property 4) leads us to eliminate **up** from the "leading 1"s.

Recall the matrix in Row-Echelon Form that we just saw:

$$\left[\begin{array}{ccccc|c} \mathbf{1} & 3 & 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 9 & 2 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

In order to obtain RRE Form, we must use row replacement EROs to kill off the three entries in purple (the "7," the "4," and the "9"); we need "0"s in those positions.

PART H: GAUSS-JORDAN ELIMINATION

This is a matrix-heavy alternative to Gaussian Elimination in which we use EROs to go all the way to RRE Form.

A matrix of numbers can have infinitely many Row-Echelon Forms [that the matrix is row-equivalent to], but it has only one **unique** RRE Form.

Technical Note: The popular MATLAB ("Matrix Laboratory") software has an "rref" command that gives this unique RRE Form for a given matrix.

In fact, we can efficiently use Gauss-Jordan Elimination to help us describe the solution set of a system of linear equations with infinitely many solutions.

Example

Let's say we have a system that we begin to solve using Gaussian Elimination. Let's say we obtain the following matrix in Row-Echelon Form:

$$\left[\begin{array}{ccc|c} \mathbf{1} & -2 & 3 & 9 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Before this Part, we would stop with the matrices and write out the corresponding system.

In Gauss-Jordan Elimination, however, we're not satisfied with just any Row-Echelon Form for our final augmented matrix. We demand RRE Form.

To obtain RRE Form, we must eliminate **up** from two of the "leading **1**"s and kill off the three purple entries: the "-2" and the two "3"s. We need "0"s in those positions.

In Gaussian Elimination, we "corrected" the columns from left to right in order to preserve our good works. At this stage, however, when we eliminate **up**, we prefer to correct the columns from **right to left** so that we can take advantage of the "0"s we create along the way.

(Reminder:)

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \left[\begin{array}{ccc|c} \mathbf{1} & -2 & 3 & 9 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Use row replacement EROs to eliminate the two "3"s in the third column. Observe that we use a "leading **1**" from a lower row to kill off an entry from a higher row.

old R_2	0	1	3		5
$+(-3)\cdot R_3$	0	0	-3		-6
new R_2	**0**	**1**	**0**		$\mathbf{-1}$

old R_1	1	-2	3		9
$+(-3)\cdot R_3$	0	0	-3		-6
new R_1	**1**	$\mathbf{-2}$	**0**		**3**

New matrix:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \left[\begin{array}{ccc|c} \mathbf{1} & -2 & 0 & 3 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Now, use a row replacement ERO to eliminate the "-2" in the second column.

old R_1	1	-2	0		3
$+2\cdot R_2$	0	2	0		-2
new R_1	**1**	**0**	**0**		**1**

Observe that our "right to left" strategy has allowed us to use "0"s to our advantage.

Here is the final RRE Form:

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

We can read off our solution now!

$$\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$

Solution set: $\{(1, -1, 2)\}$

As you can see, some work has been moved from the back-substitution stage (which is now deleted) to the ERO stage.

PART I: SYSTEMS WITH INFINITELY MANY SOLUTIONS (OPTIONAL?)

Example

Solve the system: $\begin{cases} x-2y+z+5w=3 \\ 2x-4y+z+7w=5 \end{cases}$

Warning: In fact, w is often considered to be the fourth coordinate of ordered 4-tuples of the form (x, y, z, w).

Solution

The augmented matrix is:

$$\begin{matrix} R_1 \\ R_2 \end{matrix}\left[\begin{array}{cccc|c} 1 & -2 & 1 & 5 & 3 \\ \mathbf{2} & -4 & 1 & 7 & 5 \end{array}\right]$$

Let's first go to Row-Echelon Form, which is required in both Gaussian Elimination and Gauss-Jordan Elimination – that is, unless it is clear at some point that there is no solution.

We will use a row replacement ERO and use the "1" in the upper left corner to kill off the "**2**" in the lower left corner and get a "0" in there.

old R_2	2	-4	1	7		5
$+(-2)\cdot R_1$	-2	4	-2	-10		-6
new R_2	**0**	**0**	$\mathbf{-1}$	$\mathbf{-3}$		$\mathbf{-1}$

New matrix:

$$\begin{matrix} R_1 \\ R_2 \end{matrix}\left[\begin{array}{cccc|c} 1 & -2 & 1 & 5 & 3 \\ 0 & 0 & \mathbf{-1} & -3 & -1 \end{array}\right]$$

We now need a "1" where the boldfaced "$\mathbf{-1}$" is.

To obtain Row-Echelon Form, we multiply through R_2 by (-1):

$$(\text{new } R_2) \leftarrow (-1)\cdot(\text{old } R_2)$$

$$\begin{array}{c} \\ R_1 \\ R_2 \end{array}\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} \mathbf{1} & -2 & 1 & 5 & 3 \\ 0 & 0 & \mathbf{1} & 3 & 1 \end{array}\right] \\ \text{RHS} \end{array}$$

The "leading **1**"s are boldfaced.

We first observe that the system is consistent, because of the following rule:

> An augmented matrix in Row-Echelon Form corresponds to an **inconsistent** system (i.e., a system with no solution) $\Leftrightarrow$
> (if and only if) there is a "leading **1**" in the RHS.
>
> In other words, it corresponds to a **consistent** system $\Leftrightarrow$
> there are **no** "leading **1**"s in the RHS.

Warning: There is a "1" in our RHS here in our Example, but it is **not** a "leading **1**."

Each of the variables that correspond to the columns of the coefficient matrix (here, *x*, *y*, *z*, and *w*) is either a basic variable or a free variable.

A variable is called a basic variable $\Leftrightarrow$
It corresponds to a column that has a "leading **1**."

A variable is called a free variable $\Leftrightarrow$
It corresponds to a column that does **not** have a "leading **1**."

In this Example, *x* and *z* are basic variables, and *y* and *w* are free variables.

Let's say our system of linear equations is consistent.

If there are no free variables, then the system has only one solution.

Otherwise, if there is at least one free variable, then the system has infinitely many solutions.

At this point, we know that the system in our Example has infinitely many solutions.

If we want to completely describe the solution set of a system with infinitely many solutions, then we should use Gauss-Jordan Elimination and take our matrix to RRE Form. We must kill off the "1" in purple below.

$$\begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} \mathbf{1} & -2 & 1 & 5 & 3 \\ 0 & 0 & \mathbf{1} & 3 & 1 \end{array}\right] \\ \text{RHS} \end{array}$$

old R_1	1	-2	1	5		3
$+(-1)\cdot R_2$	0	0	-1	-3		-1
new R_1	**1**	**−2**	**0**	**2**		**2**

Our RRE Form:

$$\begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} \mathbf{1} & -2 & 0 & 2 & 2 \\ 0 & 0 & \mathbf{1} & 3 & 1 \end{array}\right] \\ \text{RHS} \end{array}$$

The corresponding system:

$$\begin{cases} x - 2y \quad + 2w = 2 \\ \qquad\qquad z + 3w = 1 \end{cases}$$

Now for some steps we haven't seen before.

We will parameterize (or parametrize) the free variables:

Let $y = a,$
$w = b,$
where the parameters a and b represent any pair of real numbers.

Both of the parameters are allowed to "roam freely" over the reals.

Let's rewrite our system using these parameters:

$$\begin{cases} x - 2a \quad + 2b = 2 \\ \qquad\quad z + 3b = 1 \end{cases}$$

This is a system consisting of two variables and two parameters.

We then solve the equations for the basic variables, x and z:

$$\begin{cases} x = 2 + 2a - 2b \\ z = 1 - 3b \end{cases}$$

Remember that $y = a$ and $w = b$, so we have:

$$\begin{cases} x = 2 + 2a - 2b \\ y = a \\ z = 1 - 3b \\ w = b \end{cases}$$

Note: In your Linear Algebra class (Math 254 at Mesa), you may want to line up like terms.

We can now write the solution set.

$$\left\{ \left(2+2a-2b,\ a,\ 1-3b,\ b \right) \middle| \, a \text{ and } b \text{ are real numbers} \right\}$$

<u>Comments</u>

This set consists of infinitely many solutions, each corresponding to a different pair of choices for *a* and *b*.

Some solutions:

a	b	$\Rightarrow$	(x,	y,	z,	w)
0	0	$\Rightarrow$	(2,	0,	1,	0)
-4	7	$\Rightarrow$	(-20,	-4,	-20,	7)

Because we have two parameters, the graph of the solution set is a 2-dimensional plane existing in 4-dimensional space. Unfortunately, we can't see this graph! Nevertheless, this is the kind of thinking you will engage in in your Linear Algebra class (Math 254 at Mesa)!