PART E: WHEN DOES A SYSTEM HAVE NO SOLUTION?

If we **ever** get a row of the form:

$$0 \quad 0 \quad \cdots \quad 0 \quad \Big| \quad (\text{non-0 constant}),$$

then STOP! We know at this point that the solution set is $\varnothing$.

<u>Example</u>

Solve the system: $\begin{cases} x + y = 1 \\ x + y = 4 \end{cases}$

<u>Solution</u>

The augmented matrix is:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 4 \end{array} \right]$$

We can quickly subtract R_1 from R_2. We then obtain:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 3 \end{array} \right]$$

The new R_2 implies that the solution set is $\varnothing$.

<u>Comments</u>: This is because R_2 corresponds to the equation $0 = 3$, which cannot hold true for any pair (x, y).

If we get a row of all "0"s, such as:

$$0 \quad 0 \quad \cdots \quad 0 \;\Big|\; 0,$$

then what does that imply? The story is more complicated here.

Example

Solve the system: $\begin{cases} x + y = 4 \\ x + y = 4 \end{cases}$

Solution

The augmented matrix is:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & 1 & 4 \end{array} \right]$$

We can quickly subtract R_1 from R_2. We then obtain:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 0 \end{array} \right]$$

The corresponding system is then:

$$\begin{cases} x + y = 4 \\ \quad\;\; 0 = 0 \end{cases}$$

The equation $0 = 0$ is pretty easy to satisfy. All ordered pairs (x, y) satisfy it. In principle, we could delete this equation from the system. However, we tend not to delete rows in an augmented matrix, even if they consist of nothing but "0"s. The idea of changing the size of a matrix creeps us out.

The solution set is:

$$\{(x, y) \mid x + y = 4\}$$

The system has infinitely many solutions; they correspond to all of the points on the line $x + y = 4$.

However, a row of all "0"s does **not** automatically imply that the corresponding system has infinitely many solutions.

Example

Consider the augmented matrix:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

Because of R_1, the corresponding system actually has no solution.

See Notes 7.12 for a similar example.

The augmented matrices we have seen in this Part are **not** row equivalent to any matrix of the form

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cc|c} 1 & ? & ? \\ 0 & 1 & ? \end{array} \right].$$

There was no way to get that desired form using EROs.

What form do we aim for, then?

PART F: ROW-ECHELON FORM FOR A MATRIX

If it is impossible for us to obtain the form

$$\left[\begin{array}{cccc|c} 1 & & ? & & \\ & 1 & & & ? \\ 0 & & \ddots & & \\ & & & 1 & \end{array}\right]$$

(maybe because our coefficient matrix isn't even square), then what do we aim for? We aim for row-echelon form; in fact, the above form is a special case of row-echelon form.

> Properties of a Matrix in Row-Echelon Form
>
> 1) If there are any "all-0" rows, then they must be at the bottom of the matrix.
>
> Aside from these "all-0" rows,
>
> 2) Every row must have a "1" (called a "leading 1") as its leftmost non-0 entry.
>
> 3) The "leading 1"s must "flow down and to the right."
>
> More precisely: The "leading 1" of a row must be in a column to the right of the "leading 1"s of all higher rows.

Example

The matrix below is in Row-Echelon Form:

$$\left[\begin{array}{ccccc|c} \mathbf{1} & 3 & 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 9 & 2 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The "leading **1**"s are boldfaced.
The "1" in the upper right corner is **not** a "leading 1."

PART G: REDUCED ROW-ECHELON (RRE) FORM FOR A MATRIX

This is a special case of Row-Echelon Form.

Properties of a Matrix in Reduced Row-Echelon (RRE) Form
1-3) It is in Row-Echelon form. (See Part F.)
4) Each "leading 1" has all "0"s elsewhere in its column.

Property 4) leads us to eliminate **up** from the "leading 1"s.

Recall the matrix in Row-Echelon Form that we just saw:

$$\left[\begin{array}{ccccc|c} \mathbf{1} & 3 & 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 9 & 2 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

In order to obtain RRE Form, we must use row replacement EROs to kill off the three entries in purple (the "7," the "4," and the "9"); we need "0"s in those positions.

PART H: GAUSS-JORDAN ELIMINATION

This is a matrix-heavy alternative to Gaussian Elimination in which we use EROs to go all the way to RRE Form.

A matrix of numbers can have infinitely many Row-Echelon Forms [that the matrix is row-equivalent to], but it has only one **unique** RRE Form.

Technical Note: The popular MATLAB ("Matrix Laboratory") software has an "rref" command that gives this unique RRE Form for a given matrix.

In fact, we can efficiently use Gauss-Jordan Elimination to help us describe the solution set of a system of linear equations with infinitely many solutions.

Example

Let's say we have a system that we begin to solve using Gaussian Elimination. Let's say we obtain the following matrix in Row-Echelon Form:

$$\left[\begin{array}{ccc|c} \mathbf{1} & -2 & 3 & 9 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Before this Part, we would stop with the matrices and write out the corresponding system.

In Gauss-Jordan Elimination, however, we're not satisfied with just any Row-Echelon Form for our final augmented matrix. We demand RRE Form.

To obtain RRE Form, we must eliminate **up** from two of the "leading **1**"s and kill off the three purple entries: the "-2" and the two "3"s. We need "0"s in those positions.

In Gaussian Elimination, we "corrected" the columns from left to right in order to preserve our good works. At this stage, however, when we eliminate **up**, we prefer to correct the columns from **right to left** so that we can take advantage of the "0"s we create along the way.

(Reminder:)

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \left[\begin{array}{ccc|c} \mathbf{1} & -2 & 3 & 9 \\ 0 & \mathbf{1} & 3 & 5 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Use row replacement EROs to eliminate the two "3"s in the third column. Observe that we use a "leading **1**" from a lower row to kill off an entry from a higher row.

old R_2	0	1	3		5
$+(-3)\cdot R_3$	0	0	-3		-6
new R_2	**0**	**1**	**0**		**−1**

old R_1	1	-2	3		9
$+(-3)\cdot R_3$	0	0	-3		-6
new R_1	**1**	**−2**	**0**		**3**

New matrix:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \left[\begin{array}{ccc|c} \mathbf{1} & -2 & 0 & 3 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, use a row replacement ERO to eliminate the "-2" in the second column.

old R_1	1	-2	0		3
$+2\cdot R_2$	0	2	0		-2
new R_1	**1**	**0**	**0**		**1**

Observe that our "right to left" strategy has allowed us to use "0"s to our advantage.

Here is the final RRE Form:

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

We can read off our solution now!

$$\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$

Solution set: $\{(1, -1, 2)\}$

As you can see, some work has been moved from the back-substitution stage (which is now deleted) to the ERO stage.

PART I: SYSTEMS WITH INFINITELY MANY SOLUTIONS (OPTIONAL?)

Example

Solve the system: $\begin{cases} x-2y+z+5w=3 \\ 2x-4y+z+7w=5 \end{cases}$

Warning: In fact, w is often considered to be the fourth coordinate of ordered 4-tuples of the form (x, y, z, w).

Solution

The augmented matrix is:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 5 & 3 \\ \mathbf{2} & -4 & 1 & 7 & 5 \end{array} \right]$$

Let's first go to Row-Echelon Form, which is required in both Gaussian Elimination and Gauss-Jordan Elimination – that is, unless it is clear at some point that there is no solution.

We will use a row replacement ERO and use the "1" in the upper left corner to kill off the "**2**" in the lower left corner and get a "0" in there.

old R_2	2	-4	1	7		5
$+(-2)\cdot R_1$	-2	4	-2	-10		-6
new R_2	**0**	**0**	$\mathbf{-1}$	$\mathbf{-3}$		$\mathbf{-1}$

New matrix:

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 5 & 3 \\ 0 & 0 & \mathbf{-1} & -3 & -1 \end{array} \right]$$

We now need a "1" where the boldfaced "$\mathbf{-1}$" is.

To obtain Row-Echelon Form, we multiply through R_2 by (-1):

$$(\text{new } R_2) \leftarrow (-1)\cdot(\text{old } R_2)$$

$$\begin{array}{c} \\ R_1 \\ R_2 \end{array} \left[\begin{array}{cccc|c} x & y & z & w & \\ \mathbf{1} & -2 & 1 & 5 & 3 \\ 0 & 0 & \mathbf{1} & 3 & 1 \end{array}\right] \quad \text{RHS}$$

The "leading **1**"s are boldfaced.

We first observe that the system is consistent, because of the following rule:

> An augmented matrix in Row-Echelon Form corresponds to an **inconsistent** system (i.e., a system with no solution) $\Leftrightarrow$ (if and only if) there is a "leading **1**" in the RHS.
>
> In other words, it corresponds to a **consistent** system $\Leftrightarrow$ there are **no** "leading **1**"s in the RHS.

Warning: There is a "1" in our RHS here in our Example, but it is **not** a "leading **1**."

Each of the variables that correspond to the columns of the coefficient matrix (here, *x*, *y*, *z*, and *w*) is either a basic variable or a free variable.

A variable is called a basic variable $\Leftrightarrow$
It corresponds to a column that has a "leading **1**."

A variable is called a free variable $\Leftrightarrow$
It corresponds to a column that does **not** have a "leading **1**."

In this Example, *x* and *z* are basic variables, and *y* and *w* are free variables.

Let's say our system of linear equations is consistent.

If there are no free variables, then the system has only one solution.

Otherwise, if there is at least one free variable, then the system has infinitely many solutions.

At this point, we know that the system in our Example has infinitely many solutions.

If we want to completely describe the solution set of a system with infinitely many solutions, then we should use Gauss-Jordan Elimination and take our matrix to RRE Form. We must kill off the "1" in purple below.

$$\begin{matrix} & x & y & z & w & \\ R_1 & \mathbf{1} & -2 & 1 & 5 & 3 \\ R_2 & 0 & 0 & \mathbf{1} & 3 & 1 \end{matrix}$$

$$\begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{matrix} x & y & z & w \end{matrix} \\ \left[\begin{array}{cccc|c} \mathbf{1} & -2 & 1 & 5 & 3 \\ 0 & 0 & \mathbf{1} & 3 & 1 \end{array}\right] \\ \text{RHS} \end{array}$$

old R_1	1	-2	1	5		3
$+(-1)\cdot R_2$	0	0	-1	-3		-1
new R_1	**1**	**−2**	**0**	**2**		**2**

Our RRE Form:

$$\begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{matrix} x & y & z & w \end{matrix} \\ \left[\begin{array}{cccc|c} \mathbf{1} & -2 & 0 & 2 & 2 \\ 0 & 0 & \mathbf{1} & 3 & 1 \end{array}\right] \\ \text{RHS} \end{array}$$

The corresponding system:

$$\begin{cases} x - 2y \quad + 2w = 2 \\ \qquad\qquad z + 3w = 1 \end{cases}$$

Now for some steps we haven't seen before.

We will parameterize (or parametrize) the free variables:

Let $y = a$,
$w = b$,
where the parameters a and b represent any pair of real numbers.

Both of the parameters are allowed to "roam freely" over the reals.

Let's rewrite our system using these parameters:

$$\begin{cases} x - 2a \quad + 2b = 2 \\ \qquad\quad z + 3b = 1 \end{cases}$$

This is a system consisting of two variables and two parameters.

We then solve the equations for the basic variables, x and z:

$$\begin{cases} x = 2 + 2a - 2b \\ z = 1 - 3b \end{cases}$$

Remember that $y = a$ and $w = b$, so we have:

$$\begin{cases} x = 2 + 2a - 2b \\ y = a \\ z = 1 - 3b \\ w = b \end{cases}$$

Note: In your Linear Algebra class (Math 254 at Mesa), you may want to line up like terms.

We can now write the solution set.

$$\left\{ \left(2 + 2a - 2b,\ a,\ 1 - 3b,\ b \right) \middle| \, a \text{ and } b \text{ are real numbers} \right\}$$

Comments

This set consists of infinitely many solutions, each corresponding to a different pair of choices for *a* and *b*.

Some solutions:

a	b	$\Rightarrow$	$($ x,	y,	z,	w $)$
0	0	$\Rightarrow$	(2,	0,	1,	0)
-4	7	$\Rightarrow$	(-20,	-4,	-20,	7)

Because we have two parameters, the graph of the solution set is a 2-dimensional plane existing in 4-dimensional space. Unfortunately, we can't see this graph! Nevertheless, this is the kind of thinking you will engage in in your Linear Algebra class (Math 254 at Mesa)!