SECTION 8.2: OPERATIONS WITH MATRICES

We will not discuss augmented matrices until Part G.
For now, we will simply think of a matrix as a box of numbers.

PART A: NOTATION

The matrix $A = \left[a_{ij} \right]$, meaning that A consists of entries labeled a_{ij}, where i is the row number, and j is the column number.

Example

If A is 2×2, then $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Note: a_{12} and a_{21} are not necessarily equal. If they are, then we have a symmetric matrix, which is a square matrix that is symmetric about its main diagonal. An example of a symmetric matrix is: $\begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}$

PART B: WHEN DOES $A = B$?

Two matrices (say A and B) are equal $\Leftrightarrow$
They have the same size, and they have the same numbers (or expressions) in the same positions.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

If the matrix on the left is A, then the matrix on the right is A^T ("A transpose"). For the two matrices, the rows of one are the columns of the other.

Example

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The two matrices have different sizes.

The matrix on the left is 1×2.
It may be seen as a row vector, since it consists of only 1 row.

The matrix on the right is 2×1.
It may be seen as a column vector, since it consists of only 1 column.

Observe that the matrices are transposes of each other.
Think About It: What kind of matrix is, in fact, equal to its transpose?

PART C: BASIC OPERATIONS

Matrix addition: If two or more matrices have the same size, then you add them by adding corresponding entries. If the matrices do not have the same size, then the sum is undefined.

Matrix subtraction problems can be rewritten as matrix addition problems.

Scalar multiplication: To multiply a matrix by a scalar (i.e., a real number in this class), you multiply each entry of the matrix by the scalar.

Example

If

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

B is the 2×3 zero matrix, denoted by "0" or "$0_{2\times 3}$" – it is the additive identity for the set of 2×3 real matrices. However, when we refer to "identity matrices," we typically refer to multiplicative identities, which we will discuss later.

$$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

then …

1) Find $A+B+2C$

$$A+B+2C = \underbrace{\begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Perform matrix addition.}} + \underbrace{2\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix}}_{\text{Perform scalar multiplication}}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 1 \\ -1 & 5 & -4 \end{bmatrix}$$

2) Find $A-5C$

$$A-5C = A+(-5)C$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} + (-5)\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} -5 & -10 & 0 \\ 0 & -5 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -10 & 1 \\ -1 & -2 & 17 \end{bmatrix}$$

PART D : (A ROW VECTOR) TIMES (A COLUMN VECTOR)

We will deal with this basic multiplication problem before we go on to matrix multiplication in general.

Let's say we have a row vector $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ and a column vector $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Observe that they have the same number of entries; otherwise, our product will be undefined. This is how we multiply the row vector and the column vector (in that order); the resulting product may be viewed as either a scalar or a 1×1 matrix, depending on the context of the problem:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{cases} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n & (\text{as a scalar}) \\ \text{or} & \\ \begin{bmatrix} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{bmatrix} & (\text{as a } 1 \times 1 \text{ matrix}) \end{cases}$$

In words, we add the products of corresponding entries.
This should remind you of the dot product of two vectors, which we saw in Section 6.4: Notes 6.28.

Warning: A column vector times a row vector (in that order) gives you something very different, namely an $n \times n$ matrix. We will see why in the next Part.

Example

$$\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix} = (1)(4) + (0)(5) + (3)(-1)$$
$$= 4 + 0 - 3$$
$$= 1$$

The product may also be written as $\begin{bmatrix} 1 \end{bmatrix}$.

PART E: MATRIX MULTIPLICATION (*AB*)

When multiplying matrices, we do **not** simply multiply corresponding entries, although MATLAB does have an operation for that.

Technical Definition (Optional?)

(Bear in mind that the "tricks" that we will discuss later will make all of this easier to swallow.)

Given two matrices *A* and *B*, the matrix product *AB* is defined $\Leftrightarrow$

The rows of *A* and the columns of *B* have the same "length" (i.e., number of entries).

That is: $\left(\text{the number of columns of } A\right) = \left(\text{the number of rows of } B\right)$

If *AB* is defined, then the entry in its i^{th} row and j^{th} column equals:

$$\left(\text{the } i^{\text{th}} \text{ row of } A\right) \text{ times } \left(\text{the } j^{\text{th}} \text{ column of } B\right)$$

for appropriate values of *i* and *j*.

Another way of looking at this:

If we let $C = AB$, where $C = \left[c_{ij} \right]$, then:

$$c_{ij} = \left(\text{the } i^{\text{th}} \text{ row of } A\right) \text{ times } \left(\text{the } j^{\text{th}} \text{ column of } B\right)$$

for appropriate values of *i* and *j*.

<u>Example and Tricks</u>

Consider the matrix product $\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}}_{B}$.

Is AB defined? If so, what is its size?

Let's consider the sizes of A and B:

$$\begin{matrix} A & & B \\ 2\times\boxed{3} & & \boxed{3}\times 2 \end{matrix}$$

If the two boxed "inner numbers" are equal, then AB is defined, because:

$$\left(\text{the number of columns of } A\right) = \left(\text{the number of rows of } B\right),$$

as specified in our Technical Definition.

<u>Warning</u>: The two "outer numbers" (the "2"s here) need not be equal. The fact that they are means that the matrix will be square.

The size of AB is given by the two "outer numbers" in order.

Here, AB will be 2×2.

In general, if A is $m\times\boxed{n}$, and B is $\boxed{n}\times p$, then AB will be $m\times p$.

The trick coming up will help explain why.

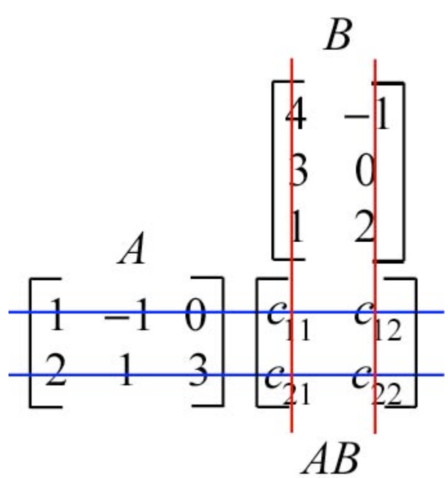
Find AB.

We will use a "traffic intersection" model.

To begin our trick, we will write B to the "northeast" (i.e., entirely above and to the right) of A.

Draw thin lines (in blue below) through the rows of A and thin lines (in red below) through the columns of B so that all intersection points are shown. These intersection points correspond to the entries of AB. We can see immediately that the size of AB will be 2×2.

Warning: Make sure your lines are thin and are placed so that, for example, no "$-$" signs or "1"s are written over.



At each intersection point, we take the corresponding row of A and the corresponding column of B, and we multiply them as we did in Part D. (We are essentially taking the dot product of the two vectors whose lines intersect at that point.)

$$c_{11} = (1)(4) + (-1)(3) + (0)(1) = 4 - 3 + 0 = \mathbf{1}$$
$$c_{12} = (1)(-1) + (-1)(0) + (0)(2) = -1 + 0 + 0 = \mathbf{-1}$$
$$c_{21} = (2)(4) + (1)(3) + (3)(1) = 8 + 3 + 3 = \mathbf{14}$$
$$c_{22} = (2)(-1) + (1)(0) + (3)(2) = -2 + 0 + 6 = \mathbf{4}$$

Therefore, $AB = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{14} & \mathbf{4} \end{bmatrix}$.

If the rows of A do not have the same length as the columns of B (so that dot products cannot be taken), then the matrix product AB is undefined.

Observe that this is all consistent with the Technical Definition.

Even though you may not have to use the c_{ij} notation, it may be a good idea to show some work for partial credit purposes.

Warning: Matrix multiplication is **not commutative**. It is often the case that $AB \neq BA$. In fact, one product may be defined, while the other is not.

Think About It: When are AB and BA both defined?

Why was matrix multiplication defined in this way? The answer lies in your Linear Algebra course (Math 254 at Mesa). The idea of "compositions of linear transformations" is key. You'll see.

PART F: IDENTITY MATRICES "*I*"

An identity matrix I is a square matrix with the following form:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{bmatrix}$$

We require "1"s along the main diagonal and "0"s everywhere else.

Observe that this is a special RRE Form; this will come up in Part G.

More specifically, I_n is the $n \times n$ identity.

<u>Example</u>

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These matrices are **multiplicative** identities. They play the role that "1" does in the set of real numbers. (Remember that zero matrices were **additive** identities.)

If A is an $n \times n$ matrix, then $AI_n = A$, and $I_n A = A$.

<u>Comment</u>: Although matrix multiplication is not, in general, commutative, it is true that an identity matrix "commutes" with a matrix of the same size. You get the same product, regardless of the order of multiplication.

<u>Note</u>: Even if A is not $n \times n$, it is possible that AI_n or $I_n A$ is defined, in which case the result is A. For example:

$$\begin{bmatrix} 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \end{bmatrix}$$

PART G: MATRIX NOTATION and SYSTEMS OF LINEAR EQUATIONS

Example (#56 on p.570)

Consider the system: $\begin{cases} x_1 + x_2 - 3x_3 = 9 \\ -x_1 + 2x_2 = 6 \\ x_1 - x_2 + x_3 = -5 \end{cases}$

Observe that this is a **square** system of linear equations; the number of equations (3) equals the number of unknowns (3).

We can write this system as a matrix (or matrix-vector) equation, $AX = B$:

$$\underbrace{\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 9 \\ 6 \\ -5 \end{bmatrix}}_{B}$$

A is the coefficient matrix (and it is square), and *B* is the RHS.
X may be thought of as a vector of variables or as a solution vector.

We will use Gauss-Jordan Elimination on the augmented matrix $\left[A \mid B \right]$ to solve for *X*. This is what we basically did in Section 8.1.

In order to save time, we will skip the steps that take us to Row-Echelon Form. Don't do this in your own work, though!

The $\sim$ symbol indicates row-equivalence, not equality.

$$\left[A \mid B \right] = \left[\begin{array}{ccc|c} 1 & 1 & -3 & 9 \\ -1 & 2 & 0 & 6 \\ 1 & -1 & 1 & -5 \end{array} \right]$$

$$\vdots$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -3 & 9 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

There are actually infinitely many possible Row-Echelon Forms for $\left[A \middle| B \right]$; the last matrix is just one of them. However, there is only one RRE Form. Let's find it. We proceed with Gauss-Jordan Elimination by eliminating up from the "leading **1**"s.

$$\left[\begin{array}{ccc|c} \mathbf{1} & 1 & -3 & 9 \\ 0 & \mathbf{1} & -1 & 5 \\ 0 & 0 & \mathbf{1} & -2 \end{array}\right]$$

We eliminate up the third column:

old R_2	0	**1**	-1		5
$+R_3$	0	0	1		-2
new R_2	**0**	**1**	**0**		**3**

old R_1	1	1	-3		9
$+3 \cdot R_3$	0	0	3		-6
new R_1	**1**	**1**	**0**		**3**

New matrix:

$$\left[\begin{array}{ccc|c} \mathbf{1} & 1 & 0 & 3 \\ 0 & \mathbf{1} & 0 & 3 \\ 0 & 0 & \mathbf{1} & -2 \end{array}\right]$$

We eliminate up the second column:

old R_1	1	1	0		3
$+(-1) \cdot R_2$	0	-1	0		-3
new R_1	**1**	**0**	**0**		**0**

RRE Form:

$$\left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 3 \\ 0 & 0 & \mathbf{1} & -2 \end{array}\right]$$

Observe that the coefficient matrix is I_3, the 3×3 identity.

The corresponding system is pretty nifty:

$$\begin{cases} x_1 = 0 \\ x_2 = 3 \\ x_3 = -2 \end{cases}$$

This immediately gives us our solution vector, X:

$$X = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

In general, if $AX = B$ represents a square system of linear equations that has exactly one solution, then the RRE Form of $\left[\, A \,|\, B \,\right]$ will be $\left[\, I \,|\, X \,\right]$, where I is the identity matrix that is the same size as A. We simply grab our solution from the new RHS.