SECTION 8.3: THE INVERSE OF A SQUARE MATRIX

PART A: (REVIEW) THE INVERSE OF A REAL NUMBER

If a is a nonzero real number, then $aa^{-1} = a\left(\frac{1}{a}\right) = 1$.

a^{-1}, or $\frac{1}{a}$, is the multiplicative inverse of a, because its product with a is 1, the multiplicative identity.

Example

$3\left(\frac{1}{3}\right) = 1$, so 3 and $\frac{1}{3}$ are multiplicative inverses of each other.

PART B: THE INVERSE OF A SQUARE MATRIX

If A is a square $n \times n$ matrix, sometimes there exists a matrix A^{-1} ("A inverse") such that

$$AA^{-1} = I_n \text{ and } A^{-1}A = I_n.$$

An invertible matrix and its inverse commute with respect to matrix multiplication.

Then, A is invertible (or nonsingular), and A^{-1} is unique.

In this course, an invertible matrix is assumed to be square.

Technical Note: A nonsquare matrix may have a left inverse matrix or a right inverse matrix that "works" on one side of the product and produces an identity matrix. They cannot be the same matrix, however.

PART C : FINDING A^{-1}

We will discuss a shortcut for 2×2 matrices in Part F.

Assume that A is a given $n \times n$ (square) matrix.

A is invertible $\Leftrightarrow$ Its RRE Form is the identity matrix I_n (or simply I).

It turns out that a sequence of EROs that takes you from an invertible matrix A down to I will also take you from I down to A^{-1}. (A good Linear Algebra book will have a proof for this.) We can use this fact to efficiently find A^{-1}.

We construct $\left[A \mid I \right]$. We say that A is in the "left square" of this matrix, and I is in the "right square."

We apply EROs to $\left[A \mid I \right]$ until we obtain the RRE Form $\left[I \mid A^{-1} \right]$. That is, as soon as you obtain I in the left square, you grab the matrix in the right square as your A^{-1}.

If you **ever** get a row of "0"s in the left square, then it will be impossible to obtain $\left[I \mid A^{-1} \right]$, and A is noninvertible (or singular).

Example

Let's go back to our A matrix from Section 8.2: Notes 8.44.

$$A = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Find A^{-1}.

Solution

We construct $\left[\, A \mid I \,\right]$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

We perform Gauss-Jordan Elimination to take the left square down to I. The right square will be affected in the process, because we perform EROs on **entire** rows "all the way across."

We will show a couple of row replacement EROs, and then we will leave the remaining steps to you.

We will kill off the purple entries and put "0"s in their places.

old R_2	-1	2	0		0	1	0
$+R_1$	1	1	-3		1	0	0
new R_2	**0**	**3**	$\mathbf{-3}$		**1**	**1**	**0**

old R_3	1	-1	1		0	0	1
$+(-1)\cdot R_1$	-1	-1	3		-1	0	0
new R_3	**0**	$\mathbf{-2}$	**4**		$\mathbf{-1}$	**0**	**1**

New matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 3 & -3 & 1 & 1 & 0 \\ 0 & -2 & 4 & -1 & 0 & 1 \end{array}\right]$$

(Your turn! Keep going ….)

RRE Form:

(Remember, this form is unique.)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & 1 \\ 0 & 1 & 0 & 1/6 & 2/3 & 1/2 \\ \underbrace{0 \quad 0 \quad 1}_{I} & & & \underbrace{-1/6 \quad 1/3 \quad 1/2}_{A^{-1}} & & \end{array}\right]$$

Check. (Optional)

You can check that $AA^{-1} = I$. If that holds, then it is automatically true that $A^{-1}A = I$. (The right inverse and the left inverse of an invertible matrix must be the same. An invertible matrix must commute with its inverse.)

PART D: USING INVERSES TO SOLVE SYSTEMS

In Section 8.2: Notes 8.44, we expressed a system in the matrix form $AX = B$:

$$\underbrace{\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 9 \\ 6 \\ -5 \end{bmatrix}}_{B}$$

A should look familiar. In Part C, we found its inverse, A^{-1}.

How can we express X directly in terms of A and B?

Review from Algebra I (Optional)

Let's say we want to solve $ax = b$, where $a \neq 0$ and a and b are real constants.

$$\begin{aligned} ax &= b \\ \underbrace{\left(\frac{1}{a}\right) a}_{=1} x &= \left(\frac{1}{a}\right) b \\ x &= \frac{b}{a} \end{aligned}$$

Because a^{-1} represents the multiplicative inverse of a, we can say that $a^{-1} = \frac{1}{a}$, and the steps can be rewritten as follows:

$$\begin{aligned} ax &= b \\ \underbrace{a^{-1} a}_{=1} x &= a^{-1} b \\ x &= a^{-1} b \end{aligned}$$

Solving the Matrix Equation $AX = B$

Assume that A is invertible.

Note: Even though 0 is the only real number that is noninvertible (in a multiplicative sense), there are many matrices other than zero matrices that are noninvertible.

It is assumed that A, X, and B have "compatible" sizes. That is, AX is defined, and AX and B have the same size.

The steps should look familiar:

$$AX = B$$

$$\underbrace{A^{-1}A}_{=I}\, X = A^{-1}B$$

$$X = A^{-1}B$$

The Inverse Matrix Method for Solving a System of Linear Equations

If A is invertible, then the system $AX = B$ has a unique solution given by $X = A^{-1}B$.

Comments

• We must left multiply both sides of $AX = B$ by A^{-1}. If we were to right multiply, then we would obtain $AXA^{-1} = BA^{-1}$; both sides of that equation are undefined, unless A is 1×1. Remember that matrix multiplication is not commutative. Although $x = ba^{-1}$ would have been acceptable in our Algebra I discussion (because multiplication of real numbers is commutative), $X = BA^{-1}$ would be inappropriate here.

• I is the identity matrix that is the same size as A. It plays the role that "1" did in our Algebra I discussion, because 1 was the multiplicative identity for the set of real numbers.

• This result is of more theoretical significance than practical significance. The Gaussian Elimination (with Back-Substitution) method we discussed earlier is often more efficient than this inverse-based process. However, $X = A^{-1}B$ is good to know if you're using software you're not familiar with. Also, there's an important category of matrices called orthogonal matrices, for which $A^{-1} = A^T$; this makes matters a whole lot easier, since A^T is trivial to find.

Back to Our Example

We will solve the system from Section 8.2: Notes 8.44 using this Inverse Matrix Method.

$$\underbrace{\begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 9 \\ 6 \\ -5 \end{bmatrix}}_{B}$$

Solution

It helps a lot that we've already found A^{-1} in Part C; that's the bulk of the work.

$$\begin{aligned} X &= A^{-1}B \\ &= \begin{bmatrix} 1/3 & 1/3 & 1 \\ 1/6 & 2/3 & 1/2 \\ -1/6 & 1/3 & 1/2 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \end{aligned}$$

This agrees with our result from the Gauss-Jordan Elimination method we used in Section 8.2: Notes 8.44-8.46.

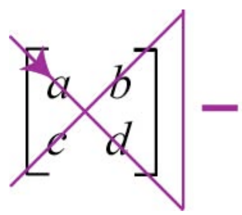
PART E : THE DETERMINANT OF A 2×2 MATRIX ("BUTTERFLY RULE")

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant of A, denoted by $\det(A)$ or $|\, A \,|$, is given by:

$$\det(A) = ad - bc$$

i.e., $\det(A) = (\text{product along main diagonal}) - (\text{product along skew diagonal})$

The following "butterfly" image may help you recall this formula.



Warning: $|\, A \,|$ should not be confused with absolute value notation.
See Section 8.4.

We will further discuss determinants in Section 8.4.

PART F : SHORTCUT FORMULA FOR THE INVERSE OF A 2×2 MATRIX

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then:

$$A^{-1}=\frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $\det(A)=0$, then A^{-1} does not exist.

Remember that we:

Switch the entries along the main diagonal.

Flip the signs on (i.e., take the opposite of) the entries along the skew diagonal.

This formula is consistent with the method from Part C.

Example

If $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find A^{-1}.

Solution

First off:

$$\begin{aligned}\det(A)&=(1)(4)-(2)(3)\\&=-2\end{aligned}$$

Now:

$$\begin{aligned}A^{-1}&=\frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\\&=\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\end{aligned}$$