SECTION 8.4: THE DETERMINANT OF A SQUARE MATRIX

PART A: INTRO

Every square matrix consisting of scalars (for example, real numbers) has a determinant, denoted by $\det(A)$ or $|A|$, which is also a scalar.

PART B: SHORTCUTS FOR COMPUTING DETERMINANTS

(We will discuss a general method in Part C. The shortcuts described here for small matrices may be derived from that method.)

1×1 Matrices

If $A = \left[c \right]$, then $\det(A) = c$.

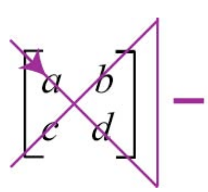
Warning: It may be confusing to write $|A| = c$. Don't confuse determinants (which **can** be negative in value) with absolute values (which **cannot**).

2×2 Matrices ("Butterfly Rule")

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = ad - bc$.

i.e., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. (Brackets are typically left out.)

We discussed this case in Section 8.3: Notes 8.55.



3×3 Matrices ("Sarrus's Rule," named after George Sarrus)

If A is 3×3, then, to find $\det(A)$:

1) Rewrite the 1st and 2nd columns on the right (as "Columns 4 and 5").
2) **Add** the products along the three full diagonals that extend from upper left to lower right.
3) **Subtract** the products along the three full diagonals that extend from lower left to upper right.

The wording above is admittedly awkward. Look at this Example:

Example

Let $A = \begin{bmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$. Find $\det(A)$.

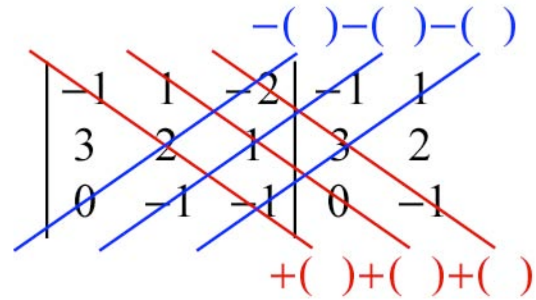
i.e., Find $\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$.

Solution

We begin by rewriting the 1st and 2nd columns on the right.

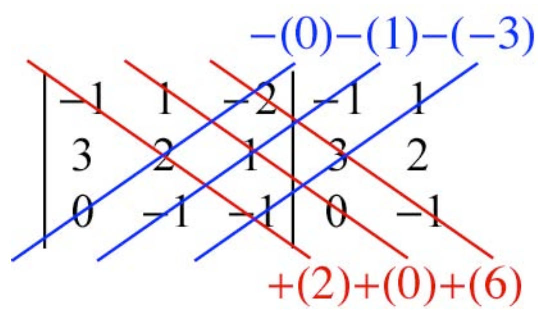
$$\left|\begin{array}{ccc|cc} -1 & 1 & -2 & -1 & 1 \\ 3 & 2 & 1 & 3 & 2 \\ 0 & -1 & -1 & 0 & -1 \end{array}\right.$$

In order to avoid massive confusion with signs, we will set up a template that clearly indicates the products that we will add and those that we will subtract.



The "product along a [full] diagonal" is obtained by multiplying together the three numbers that lie along the diagonal. We will compute the six products corresponding to our six indicated diagonals, place them in the parentheses in our template, and compute the determinant.

<u>Time-Saver</u>: If a diagonal contains a "0," then the corresponding product will automatically be 0.



Therefore,

$$\begin{aligned}\det(A) &= 2+0+6-0-1+3 \\ &= 10\end{aligned}$$

<u>Warning</u>: Although Sarrus's Rule seems like an extension of the Butterfly Rule from the 2×2 case, there is no similar shortcut algorithm for finding determinants of 4×4 and larger matrices. Sarrus's Rule is, however, related to the "permutation-based" definition of a determinant, which you may see in an advanced class.

PART C: "EXPANSION BY COFACTORS" METHOD FOR COMPUTING DETERMINANTS

This is hard to explain without an Example to lean on!

This method works for square matrices of **any** size.

Example

Find $\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$.

(In Part B, we already found out this equals 10.)

Solution

Choose a "magic row or column" to expand along, preferably one with "0"s. We will call its entries our magic entries.

In principle, you could choose any row or any column.
Here, let's choose the 1st column, in part because of the "0" in the lower right corner.

$$\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$$

Because we are dealing with a 3×3 matrix, we will set up the 3×3 sign matrix. This is always a "checkerboard" matrix that begins with a "+" sign in the upper left corner and then alternates signs along rows and columns.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

We really only need the signs corresponding to our magic row or column.

Note: The sign matrix for a 4×4 matrix is given below.

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

Technical Note: The sign of the (i, j) entry of the sign matrix is the sign of $(-1)^{i+j}$, where i is the row number of the entry, and j is the column number.

Back to our Example:

$$\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} \text{ with sign matrix } \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The following may be confusing until you see it "in action" on the next page.

Our cofactor expansion for the determinant will consist of three terms that correspond to our three magic entries. Each term will have the form:

(Sign from sign matrix) (Magic entry) (Corresponding minor),

where the "corresponding minor" is the determinant of the submatrix that is obtained when the row and the column containing the magic entry are deleted from the original matrix.

Note: The "corresponding cofactor" is the same as the corresponding minor, except that you incorporate the corresponding sign from the sign matrix. In particular, if the corresponding sign is a "$-$" sign, then the cofactor is the opposite of the minor. Then, the determinant is given by the sum of the products of the magic entries with their corresponding cofactors.

Here, we have:

$$\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} = +(-1)\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$$

$$-\ (3)\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}$$

$$+\ (0)\begin{vmatrix} \text{Who} \\ \text{cares?} \end{vmatrix}$$

Observe that the third minor is irrelevant, because we know that the third term will be 0, anyway. This is why we like choosing magic rows and columns that have "0"s in them!

There are various ways to write out the cofactor expansion quickly and accurately. With practice, you should find the one that works best for you. Some people may need to write out the step above.

We now have:

$$\begin{vmatrix} -1 & 1 & -2 \\ 3 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix} = +(-1)\underbrace{\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}}_{\substack{=-2-(-1) \\ =-1}}$$

$$-\ (3)\underbrace{\begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix}}_{\substack{=-1-(2) \\ =-3}}$$

$$\begin{aligned} &= (-1)(-1) - 3(-3) \\ &= 10 \end{aligned}$$

Note 1: It is a coincidence that the magic entries -1 and 3 are equal to their corresponding cofactors here.

Note 2: Observe that we got the same answer when we used Sarrus's Rule back in Part B. We better have!

Note 3: Observe that we expand the determinant of a 3×3 matrix in terms of the determinants of up to three 2×2 matrices. Likewise, we expand the determinant of a 4×4 matrix in terms of the determinants of up to four 3×3 matrices. This is why we like exploiting "0"s along a magic row or column – and why it is often painful to compute determinants of large matrices using this cofactor expansion method.

Note 4: An efficient alternative method employs the EROs we discussed back in Section 8.1 on Gaussian Elimination:

- Row Replacement EROs preserve determinants.

  For example,

$$\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

- A single Row Interchange (Switch) ERO flips the sign of the determinant.

  For example,

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

- When computing determinants, a nonzero scalar may be "factored out" of an entire row or an entire column.

  For example,

$$\begin{vmatrix} 4 & 8 \\ 7 & 9 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 \\ 7 & 9 \end{vmatrix}$$

Note 5: The following basic determinant properties are useful, particularly in the Gaussian Elimination method for computing determinants:

- If a square matrix has a row or a column consisting of all "0"s, then its determinant is 0.

For example,

$$\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

- If a square matrix is in triangular form (i.e., has all "0"s above or below the main diagonal), then its determinant equals the product of the entries along the main diagonal.

For example,

$$\begin{vmatrix} 2 & 70 & 30 \\ 0 & 3 & 50 \\ 0 & 0 & 4 \end{vmatrix} = (2)(3)(4) = 24$$

Can you see how the above properties are derived from the Cofactor Expansion method for computing determinants?

Example

Find $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$.

Solution

Remember that there is no nice analog to Sarrus's Rule here, because we are dealing with a 4×4 matrix.

Let's expand along the 2nd row so that we can exploit its "0"s.

We have:

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} \text{ with partial sign matrix } \begin{vmatrix} + & - & + & \\ & & - & \\ & & & \\ & & & \end{vmatrix}$$

Observe that, as far as the sign matrix goes, we only need to know that the "$-$" sign corresponds to the magic "3." To find this out, you could either start with the "+" in the upper left corner and snake your way to that position (see above), or you could observe that the "3" is in Row 2, Column 3, and $(-1)^{2+3} = (-1)^5 = -1$.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = -(3) \underbrace{\begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}}_{\substack{\text{Use Sarrus's Rule or} \\ \text{Cofactor Expansion.} \\ \text{It turns out this equals 2.} \\ \text{You show work!}}}$$

$$= -3(2)$$

$$= -6$$

PART D : THE CROSS PRODUCT OF TWO VECTORS IN $\mathbf{R}^3$

In Section 6.4, we discussed the dot product of two vectors in $\mathbf{R}^n$ (n-dimensional real space).

There is another common way to multiply two vectors in $\mathbf{R}^3$ (3-dimensional real space), specifically.

Given two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in $\mathbf{R}^3$, the cross product $\mathbf{a} \times \mathbf{b}$ is given by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are the standard unit vectors in $\mathbf{R}^3$.

This notation is informal, because the determinant is only "officially" defined if our matrix consists only of scalars.

Note: Although the dot product operation is commutative (i.e., $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ for two vectors $\mathbf{a}$ and $\mathbf{b}$ in the same space), the cross product operation is not. In fact, the cross product operation is anticommutative, meaning that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$. Recall from Notes 8.63 that:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Geometrically, $\mathbf{a} \times \mathbf{b}$ is a vector that is perpendicular (or orthogonal) to both $\mathbf{a}$ and $\mathbf{b}$.