SECTION 8.5: APPLICATIONS OF DETERMINANTS

PART A: CRAMER'S RULE FOR SOLVING SYSTEMS

A square system of linear equations is a system of n linear equations in n unknowns, where $n \in \mathbf{Z}^+$. Cramer's Rule uses determinants to solve such a system. For now, we assume that the unknowns are x, y, etc. and that they make up X, the vector of unknowns.

Cramer's Rule

Write the augmented matrix for the system $AX = B$:

$$\left[A \mid B \right]$$

- A is the coefficient matrix.

  If the system is square, A will be a square matrix.

- B is the right-hand side (*RHS*); you could use *RHS*, instead.

Compute the following determinants:

- Let $D = \left| A \right|$, or $\det\left(A\right)$.
- Let $D_x = \left| A_x \right|$, or $\det\left(A_x\right)$.

  where A_x is identical to A, except that the x-column of A is replaced by B, the *RHS*.

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Cramer's Rule (cont.)

- Let $D_y = \left| A_y \right|$, or $\det\left(A_y\right)$,

  where A_y is identical to *A*, except that the *y*-column of *A* is replaced by *B*, the *RHS*.

- D_z, A_z, etc. are defined analogously as necessary.

If $D \neq 0$, **there is a unique solution** given by:

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D} \text{ (if applicable), etc.}$$

If $D = 0$, there is **not** a unique solution. Then:

- If all of the other determinants, D_x, D_y, etc. are also 0, then the system has infinitely many solutions.

- Otherwise, the system has no solution. The solution set is $\varnothing$, the empty set.

Note: Observe that the formulas for *x*, *y*, etc. fall apart if $D = 0$.

Note: In fact, if *A* is square, then its determinant $D \neq 0$ if and only if *A* is invertible, which is true if and only if $AX = B$ has a unique solution (given by $X = A^{-1}B$). See the Inverse Matrix Method for solving systems in Section 8.3, Part D.

Note: One advantage that this method has over Gaussian Elimination with Back-Substitution is that the value of one unknown can be found without having to find the values of any others.

Technical Note: For large systems, the Expansion by Cofactors Method for computing determinants (found in Section 8.4, Part C) may be impractical. See Notes 3, 4, and 5 in Notes 8.63 and 8.64.

Example (Two linear equations in two unknowns)

Solve the system: $\begin{cases} 2x-9y= \ 5 \\ 3x-3y=11 \end{cases}$

Solution

The augmented matrix is:

$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 2 & -9 & 5 \\ 3 & -3 & 11 \end{array}\right] \end{array} \quad \left[\begin{array}{c|c} \underbrace{\begin{matrix} 2 & -9 \\ 3 & -3 \end{matrix}}_{A} & \underbrace{\begin{matrix} 5 \\ 11 \end{matrix}}_{B} \end{array}\right]$$

Compute the necessary determinants:

Note: Your instructor may want you to show more work here.

$$D = \left| A \right| = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{vmatrix} 2 & -9 \\ 3 & -3 \end{vmatrix} \end{array} = 21$$

Warning: When constructing the A_x and A_y matrices, which are "mutated" versions of the *A* matrix, remember to replace the correct column with *B*, the *RHS*. You replace the column corresponding to the subscript, which is the variable that the matrix helps solve for. See the Warning in the next Example.

$$D_x = \left| A_x \right| = \begin{array}{c} \begin{array}{cc} B & y \end{array} \\ \begin{vmatrix} 5 & -9 \\ 11 & -3 \end{vmatrix} \end{array} = 84$$

$$D_y = \left| A_y \right| = \begin{array}{c} \begin{array}{cc} x & B \end{array} \\ \begin{vmatrix} 2 & 5 \\ 3 & 11 \end{vmatrix} \end{array} = 7$$

Since $D \neq 0$, the system has a unique solution, which is given by:

$$x = \frac{D_x}{D} = \frac{84}{21} = 4$$

$$y = \frac{D_y}{D} = \frac{7}{21} = \frac{1}{3}$$

Warning: You may have been tempted to write down the fraction $\frac{21}{7}$. Remember that non-integers may appear in your solutions.

The solution set is then: $\left\{\left(4, \frac{1}{3}\right)\right\}$.

Note: Our solution may be checked in the original system.

Note: Observe that we can solve for x without solving for y.

Example (Three linear equations in three unknowns)

Solve the system: $$\begin{cases} x \phantom{{}-3y} + z = 0 \\ x - 3y \phantom{{}+z} = 1 \\ 4y - 3z = 3 \end{cases}$$

Solution

The augmented matrix is:

$$\begin{array}{ccc|c} x & y & z & \\ \hline \end{array}$$

$$\left[\begin{array}{ccc|c} x & y & z & \\ 1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 1 \\ \underbrace{0 \quad 4 \quad -3}_{A} & & & \underbrace{3}_{B} \end{array}\right]$$

Compute the necessary determinants:

Note: Your instructor may want you to show more work here.

$$D = |A| = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{vmatrix} 1 & 0 & 1 \\ 1 & -3 & 0 \\ 0 & 4 & -3 \end{vmatrix} \end{array} = 13$$

Warning: When constructing the A_x matrix, remember to replace the x-column with B, the RHS, and leave the y- and z-columns intact. (If you remember this, then the two variable case may be less confusing.) The A_y and A_z matrices are constructed analogously.

$$D_x = |A_x| = \begin{array}{c} \begin{array}{ccc} B & y & z \end{array} \\ \begin{vmatrix} 0 & 0 & 1 \\ 1 & -3 & 0 \\ 3 & 4 & -3 \end{vmatrix} \end{array} = 13$$

$$D_y = \left| A_y \right| = \begin{array}{c} \begin{array}{ccc} x & B & z \end{array} \\ \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 3 & -3 \end{vmatrix} \end{array} = 0$$

$$D_z = \left| A_z \right| = \begin{array}{c} \begin{array}{ccc} x & y & B \end{array} \\ \begin{vmatrix} 1 & 0 & 0 \\ 1 & -3 & 1 \\ 0 & 4 & 3 \end{vmatrix} \end{array} = -13$$

Since $D \neq 0$, the system has a unique solution, which is given by:

$$x = \frac{D_x}{D} = \frac{13}{13} = 1$$

$$y = \frac{D_y}{D} = \frac{0}{13} = 0$$

$$z = \frac{D_z}{D} = \frac{-13}{13} = -1$$

The solution set is then: $\left\{ \left(1, 0, -1 \right) \right\}$.

<u>Note</u>: Our solution may be checked in the original system.

PART B: AREA AND VOLUME

In Calculus: In Multivariable Calculus (Calculus III: Math 252 at Mesa), you may study triple scalar products (when dealing with three-dimensional vectors) and Jacobians, which employ the following ideas.

Determinants and Area

Assume that A is a 2×2 matrix of real numbers. Consider the position vectors corresponding to either the rows or the columns of A.

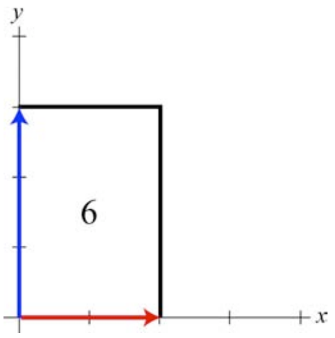
The area of the parallelogram determined by those vectors is given by $\left|\det(A)\right|$, or $\left|\,|A|\,\right|$, the absolute value of the determinant of A.

(If it is 0, the vectors are collinear – they lie on the same line, and the parallelogram is degenerate.)

In these Examples, we will consider the position vectors corresponding to the **columns** of the matrices.

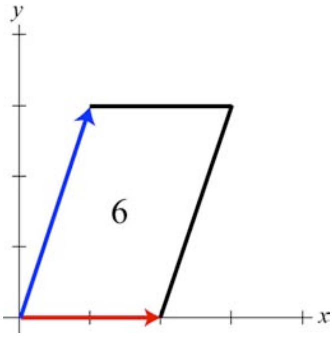
Example

$$\left\| \begin{matrix} 2 & 0 \\ 0 & 3 \end{matrix} \right\| = 6$$



Example

$$\left\| \begin{matrix} 2 & 1 \\ 0 & 3 \end{matrix} \right\| = 6$$

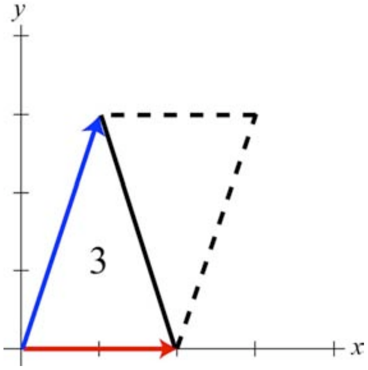


Think About It: Can you give other reasons why these parallelograms have the same area?

Follow-Up Example

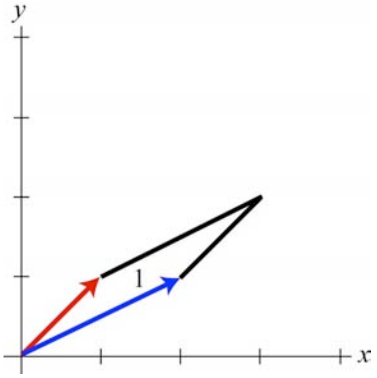
The area of the **triangle** determined by the position vectors of interest equals **half** the area of the **parallelogram** determined by them.

$$\frac{1}{2}\left|\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix}\right| = \frac{1}{2}(6) = 3$$



Example

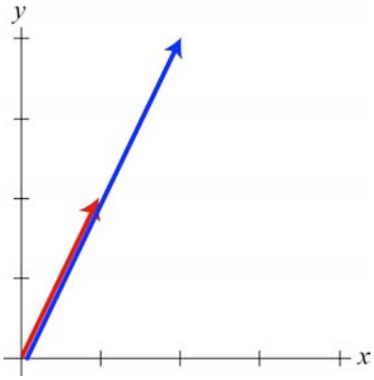
$$\left|\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}\right| = |-1| = 1$$



Think About It: Why do you think the determinant is negative in this Example? Test your guess by trying out some examples of your own.

Example

$$\left\| \begin{matrix} 1 & 2 \\ 2 & 4 \end{matrix} \right\| = 0$$



The position vectors here are collinear.

Technical Note: We may analyze the row vectors or the column vectors of the matrix for the purposes of finding area or volume, because a square matrix and its transpose (see Notes 8.35) have the same determinant. i.e., If A is a square matrix, then $\det(A) = \det(A^T)$.

Technical Note: If the rows or the columns of a square matrix are reordered, then the determinant will change by at most a sign, and its absolute value stays the same. Therefore, the row or column vectors may be written in the matrix in any order for the purposes of finding area or volume.

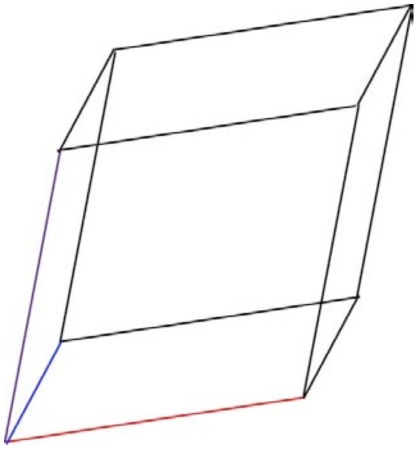
Determinants and Volume

Assume that A is a 3×3 matrix of real numbers. Consider the position vectors corresponding to either the rows or the columns of A.

The volume of the parallelepiped determined by those vectors is given by $\left|\det(A)\right|$, or $\left|\,|A|\,\right|$, the absolute value of the determinant of A.

(If it is 0, the vectors are coplanar – they lie on the same plane, and the parallelepiped is degenerate.)

A parallelepiped:



In Calculus: In Multivariable Calculus (Calculus III: Math 252 at Mesa), you may study the triple scalar product ("TSP") of the row or column vectors ($\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, say) in the 3×3 matrix A. The TSP equals $\det(A)$. The TSP can also be written, and is usually defined, in terms of dot and cross products as: $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$, or $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$. For more information, see my Math 252 notes on Section 14.4 in the Swokowski Calculus text.