

CHAPTER 9: DISCRETE MATH

Calculus tends to deal more with “continuous” mathematics than “discrete” mathematics. What is the difference? Analogies may help the most. Discrete is digital; continuous is analog. Discrete is a dripping faucet; continuous is running water. Discrete math tends to deal with things that you can “list,” even if the list is infinitely long. [Math 245 is the Discrete Math course at Mesa.](#)

SECTION 9.1: SEQUENCES AND SERIES, and SECTION 9.6: COUNTING PRINCIPLES

PART A: SEQUENCES

An infinite sequence can usually be thought of as an ordered list of real numbers. It is often denoted by either:

$$a_1, a_2, a_3, \dots$$

or

$$a_0, a_1, a_2, \dots \quad (\text{this is more common in Calculus})$$

where each of the “ a_i ”s represent real numbers called terms.

If we’re lucky, there is a nice pattern to our sequence of numbers – a pattern that we can describe using simple mathematical expressions.

Example

Write the first three terms of the sequence with general n^{th} term $a_n = n^3 - 1$.

Assume that we begin with $n = 1$.

Solution

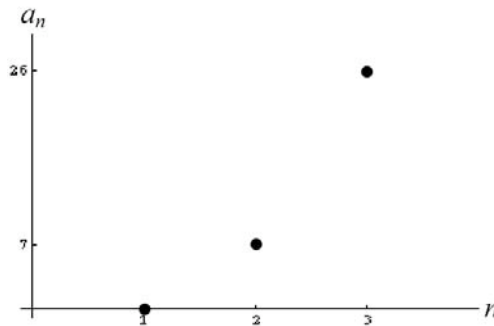
$$n = 1: a_1 = (1)^3 - 1 = 0$$

$$n = 2: a_2 = (2)^3 - 1 = 7$$

$$n = 3: a_3 = (3)^3 - 1 = 26$$

The first three terms are: 0, 7, and 26.

We may graph terms as the points (n, a_n) , just as we used to graph points $(x, f(x))$.



Technical Note: Observe that the graph of the sequence is a “discretized” version of the graph of $f(x) = x^3 - 1$, where the domain is the set of positive integers. In this sense, sequences may be thought of as functions.

PART B: SIGN ALTERNATORS

Alternating sequences have terms that alternate (i.e., consistently switch) between positive and negative terms. They arise frequently in Calculus.

Example

Write the first four terms of the sequence where $a_n = (-1)^n n$.

Assume that we begin with $n = 1$.

Solution

$$n = 1: a_1 = (-1)^1 \cdot 1 = -1$$

$$n = 2: a_2 = (-1)^2 \cdot 2 = 2$$

$$n = 3: a_3 = (-1)^3 \cdot 3 = -3$$

$$n = 4: a_4 = (-1)^4 \cdot 4 = 4$$

Recall that the parity of an integer is either “even” or “odd.”
The parity of n here determines the sign of the term.

Example

How can we obtain an alternating sequence like the one above, but starting with a positive term? Let’s adjust our sign alternator.

$$\text{Try: } a_n = (-1)^{n+1} n, \text{ or } a_n = (-1)^{n-1} n$$

In either case, we obtain: 1, -2, 3, -4, ...

PART C: EVEN VS. ODD

$2n$ yields even integers, where $n \in \mathbf{Z}$.

$2n - 1$ and $2n + 1$ yield odd integers, where $n \in \mathbf{Z}$.

These are often used to describe sequences in Calculus.

PART D: FACTORIALS

Do you see the $n!$ button on your calculator? This is n factorial (not n excited). It is often used to describe sequences.

We define:

$$0! = 1$$

If n is a positive integer ($n \in \mathbf{Z}^+$),

$n!$ = the product of the integers from 1 to n

$$\text{If } n \text{ is large: } n! = (1)(2)(3) \cdots (n)$$

$$\text{Equivalently: } n! = (n)(n-1)(n-2) \cdots (2)(1) \quad (\text{Think: Countdown.})$$

Examples

$$0! = 1$$

$$1! = 1$$

$$2! = (2)(1) = 2$$

$$3! = (3)(2)(1) = 6$$

$$4! = (4)(3)(2)(1) = 24$$

There is a short cut: $4! = 4 \cdot 3!$

Factorial is at the same level as exponentiation in the order of operations; they both represent successive multiplications. Therefore, the factorial operator (!) binds more tightly than the multiplication operator (\cdot), which is lower in the order of operations. In this case, $4! = (4) \cdot (3!)$; this may be clearer.

In general, if $n \in \mathbf{Z}^+$: $n! = n \cdot (n-1)!$, and $0! = 1$

This is called a recursive definition of the factorial function, because it describes how previous values determine later values. In this case, we show how we get from $(n-1)!$ to $n!$.

Technical Note: Can we talk about $3.5!$, say? The gamma function, which you may study in [Math 151: Calculus II at Mesa](#), is a continuous version of the factorial function.

PART E: FACTORIALS, COUNTING PRINCIPLES, AND SIMPLIFICATIONS **(SECTION 9.6)**

Example

How many ways are there to order a standard deck of 52 cards?

Solution

There are 52 possibilities for the first card.

Once the first card is set, there are 51 possibilities for the second card.

Once the first two cards are set, there are 50 possibilities for the third card.

And so on...

Once the first 51 cards are set, there is only 1 possibility left for the last card.

By the Fundamental Counting Principle described on [p.663 of Larson](#), we multiply together these numbers of possibilities.

The number of ways is:

$(52)(51)(50)\cdots(1)$, better known as $52!$, which is about 8×10^{67} .

That's an 8 followed by 67 digits – literally astronomical!

$n!$ grows in value **very** fast as n grows. Many calculators can't even handle $100!$.

In general:

The number of ways to order n distinct items is: $n!$

The different possible orders are called permutations.

Example (An elaboration on Example 9 on p.668 in Larson.)

How many possible “hands” of five cards can be drawn from a deck of 52 cards? In other words, how many ways are there to choose five cards from a deck of 52? (The order among the five chosen cards does not matter.)

Solution

This number is denoted by ${}_{52}C_5$, or $\binom{52}{5}$, called “52 choose 5.”

This is the number of combinations of 52 (distinct) items taken 5 at a time.

It turns out that $\binom{52}{5} = \frac{52!}{5!47!}$. Why?

There are $52!$ ways to order the complete deck of 52 cards. Imagine a divider separating the first five cards from the last (other) 47 cards. Let’s call the first five cards the “winners” and the last 47 cards the “losers.” We do not care about the order among the five winners, so we divide by $5!$. (The technical reason for this is somewhat subtle and may not be immediately clear.) Likewise, we do not care about the order among the 47 losers (though poker players may beg to differ), so we divide by $47!$, also.

How can we simplify and compute $\frac{52!}{5!47!}$ so that a calculator can handle it without a problem?

Good calculators should be able to handle this, but it is a good skill to know how to cancel factors when there are factorials involved.

The idea here is to unravel one or more of the factorials so that we can cancel factorials.

$$\begin{aligned} \frac{52!}{5!47!} &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5! \cancel{47!}} \\ &= 2,598,960 \end{aligned}$$

By the way, there is about a 50-50 chance of getting at least a “pair.”

Incidentally, we were guaranteed to get an integer because of the nature of the problem. This may not have been obvious from the form of the expression.

In general:

The number of ways to choose r distinct items from a set of n distinct items (if the order among the chosen items is irrelevant) is given by:

$${}_n C_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Think: The number of ways to choose r winners and, therefore, $n - r$ losers.

These $\binom{n}{r}$ numbers are called binomial coefficients for reasons we shall see in [Section 9.5](#).

See if you can find the ${}_n C_r$ button on your calculator.

What about ${}_n P_r$?

You may also see this button on your calculator. It is the number of partial permutations of 52 (distinct) items taken 5 at a time.

Let's say we actually care about the order of the five chosen cards. The number of such partial permutations is given by:

$$\begin{aligned} \frac{52!}{47!} &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!}_1} \\ &= 311,875,200 \end{aligned}$$

The idea is that we take the number of total permutations of the 52 cards (namely, $52!$), and we divide by $47!$, because we do not care about the order among the losers. This time, we do **not** divide by $5!$, because we do care here about the order among the winners.

PART F: RECURSIVELY DEFINED SEQUENCES

Recall the recursive definition of factorials: $n! = n \cdot (n-1)!$ ($n \in \mathbf{Z}^+$)

We should add that $0! = 1$, because we need to know where to start.

Likewise, we can define sequences by describing how terms are “built upon” previous terms.

Perhaps the most famous recursively defined sequence is the Fibonacci sequence (named after Leonardo of Pisa, or “Fibonacci,” a famous Italian mathematician of the Middle Ages who has been credited with introducing Arabic numerals to the West), which is described on [p.616 in Larson](#). There are surprising applications of this sequence in nature: sunflower petals, pine cone and seashell patterns, etc. You are virtually assured of dealing with this sequence in a Discrete Math class. Finding the general n^{th} term

expression is a bit of a challenge! It turns out to be: $a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$.

Yes, we get integers out of this thing! Technical details and other properties are available at the URL:

<http://mathworld.wolfram.com/FibonacciNumber.html>

Example

Consider the sequence recursively defined by:
$$\begin{cases} a_1 = 10 \\ a_{k+1} = 2a_k \end{cases} \quad (k \in \mathbf{Z}^+)$$

Find the first three terms of this sequence, beginning with $n = 1$.

Solution

We essentially double one term in the sequence to obtain the next term.

$$a_1 = 10$$

$$a_2 = 2a_1 = 2(10) = 20$$

$$a_3 = 2a_2 = 2(20) = 40$$

PART G: SUMMATION (SIGMA) NOTATION

Σ is the Greek letter known as uppercase sigma. (Lowercase sigma, σ , is famous for denoting standard deviation in statistics.) Σ is a summation operator; we use it to efficiently indicate the addition of terms.

Example

Evaluate the sum $\sum_{i=0}^3 \underbrace{(5i^2 - 1)}_{a_i}$.

Terminology and Notation

Here, i serves as the index of summation. It does not matter if we use j , k , etc., as long as there are no conflicts with other notations, and we use these notations consistently. We can think of i as a dummy variable. It's like your new co-worker – if we replaced one dummy with another, nothing really changes. **In Calculus, dummy variables are used in definite integration.**

Here, 0 is the lower limit (of summation), and 3 is the upper limit.

Solution

We need to successively replace i with the integers from 0 through 3 and then add the results.

$$\begin{aligned} \sum_{i=0}^3 a_i &= a_0 + a_1 + a_2 + a_3 \\ &= \underbrace{\left[5(0)^2 - 1\right]}_{a_0} + \underbrace{\left[5(1)^2 - 1\right]}_{a_1} + \underbrace{\left[5(2)^2 - 1\right]}_{a_2} + \underbrace{\left[5(3)^2 - 1\right]}_{a_3} \\ &= -1 + 4 + 19 + 44 \\ &= 66 \end{aligned}$$

PART H: SERIES

A series arises when we take the sum of terms of a sequence.

The following is an example of a finite series:

$$\sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \dots + a_n$$

The following is an example of an infinite series:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots$$

In fact, the finite series above represents the n^{th} partial sum of the infinite series above; it represents the sum of the first n terms.