SECTION 9.4: MATHEMATICAL INDUCTION

What is the sum of the first n positive integers, where $n \in \mathbf{Z}^+$? In other words, what is $1+2+3+...+n$? According to our formula for the n^{th} partial sum of an arithmetic sequence (see Section 9.2), the answer is: $S_n = n\left(\frac{1+n}{2}\right) = \frac{n(n+1)}{2}$

Handshake Problem (Cool, but Optional)

There's another way of seeing why this formula works out.

Imagine $n+1$ people walking into a room one-by-one. Whenever a person walks into a room, he/she must shake hands exactly once with every other person who is currently in the room, and those are the only handshakes they make. The first person who walks into the room shakes nobody's hand, the second person shakes the first person's hand, the third person shakes the first two people's hands, and so on, until the last person shakes the other n people's hands. This means that every distinct pair of people eventually shake hands exactly once. The total number of handshakes then equals the number of distinct pairs of people that can be formed from a group of $n+1$ people.

We see that the number of handshakes equals both $1+2+3+...+n$ and

$$\binom{n+1}{2} = \frac{(n+1)!}{2!\left((n+1)-2\right)!} = \frac{(n+1)!}{2!(n-1)!} = \frac{(n+1)n\cdot\cancel{(n-1)!}}{2\cdot\cancel{(n-1)!}} = \frac{n(n+1)}{2}.$$

Therefore: $1+2+3+...+n = \frac{n(n+1)}{2}$

In this section, we will use mathematical induction to prove that: $1+2+3+...+n = \frac{n(n+1)}{2}$

Mathematical induction is used to prove conjectures, statements that we believe to be true but are as yet unproven. Unfortunately, the method of mathematical induction cannot give us a formula such as $1+2+3+...+n = \frac{n(n+1)}{2}$; we must first guess at such a formula by perhaps working out a few cases and using trial-and-error. (For example, try out Problem #90 in Section 9.2 on p.633 in Larson.) Induction is then used to verify our guess.

Induction is commonly used in Discrete Math and in Linear Algebra. It is even used in continuous mathematics; in Calculus, for example, induction is used to prove integration formulas involving powers of (or products of powers of) trig functions.

The Domino Image

Visualize a half-line (or a "ray") of infinitely many dominoes. When are we guaranteed that all the dominoes will eventually fall? If we are guaranteed the following:

• The first domino falls.
(This corresponds to the Basis Step.)

• The fall of one domino guarantees the fall of the next domino.
(This corresponds to the Inductive Step.)

The Proof of Our Formula

Conjecture:

The following is true for all positive integers n $\left(\text{i.e., } \forall n \in \mathbf{Z}^{+}\right)$:

$$P_n \text{, which states that: } 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

Basis Step:

Verify that P_1 is true. (i.e., Verify that P_n is true for $n = 1$.)

$$1 = \frac{1(1+1)}{2}$$
$$1 = \frac{1(2)}{2}$$
$$1 = 1$$

This demonstrates that P_1 is true, because we have shown that P_1 is equivalent to the true statement $1 = 1$.

Note: If we only had to prove P_n for all integers *n* such that $n \geq 7$, say, then we would verify P_7 as our Basis Step.

Inductive Step:

Let *k* be any fixed positive integer. (Some books use *n*; whatever you do, be consistent throughout the problem!)

Assume that P_k is true. (This assumption is called the Inductive Hypothesis, which we will abbreviate as I.H.)

Using the Domino Image: Assume that the k^{th} domino has fallen.

Here, we assume: $1+2+3+\ldots+k=\dfrac{k(k+1)}{2}$

Show that P_{k+1} is then true.

Using the Domino Image: We must then show that the $(k+1)^{\text{st}}$ domino must then fall as a result.

Here, we must show that, as a result:

$$1+2+3+\ldots+(k+1)=\frac{(k+1)((k+1)+1)}{2}, \text{ or } \frac{(k+1)(k+2)}{2}$$

This is somewhat similar to a verification problem in trig in that the right-hand side is something of a TARGET.

$$1+2+3+\ldots+(k+1) = \underbrace{1+2+3+\ldots+k}_{\text{Apply the I.H. to this.}} + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+2)(k+1)}{2} \qquad (\text{from Factoring})$$

$$= \frac{(k+1)(k+2)}{2} \qquad (\text{TARGET})$$

Note: The argument above works even for $k=1$, $k=2$, and $k=3$, even though the first line may seem incompatible with those cases.

Conclusion

We may now write:

$\therefore$ P_n is true for all positive integers *n*.

or

P_n is true for all positive integers *n*. QED.

Note: QED stands for the Latin phrase Quod Erat Demonstrandum. It effectively means "that which was to have been proven."

Read p.648 in Larson for formulas for sums of powers of the first *n* positive integers.

Challenge Problem

Use induction to prove that every amount of postage greater than or equal to 12 cents can be formed by using just 4- and 5-cent stamps.