SECTION 9.5: THE BINOMIAL THEOREM

PART A: INTRO

How do we expand $(a+b)^n$, where n is a nonnegative integer?

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = (a+b)\underbrace{(a+b)^2}_{\text{Do first}}$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$
$$\vdots$$
$$(a+b)^{10} = \text{YUK!}$$

Look for patterns

Take $(a+b)^3$. Let's look at the variable parts of the terms.

$$(a+b)^3 = \underbrace{a^3}_{\substack{\text{Starts}\\\text{with}\\a^3}} + 3\underbrace{a^2b} + 3\underbrace{ab^2} + \underbrace{b^3}_{\substack{\text{Ends}\\\text{with}\\b^3}}$$

$$\longrightarrow \quad \longrightarrow \quad \longrightarrow$$

At each step,

- the exponent on a $\downarrow$ by 1
- the exponent on b $\uparrow$ by 1

$a^0 = 1$, $b^0 = 1$, so they are "invisible."

Each term has degree 3.

<u>In general,</u>

$$(a+b)^n = \underbrace{a^n + \ldots + b^n}_{(n+1)\text{ terms}} \quad (n \text{ is a whole number})$$

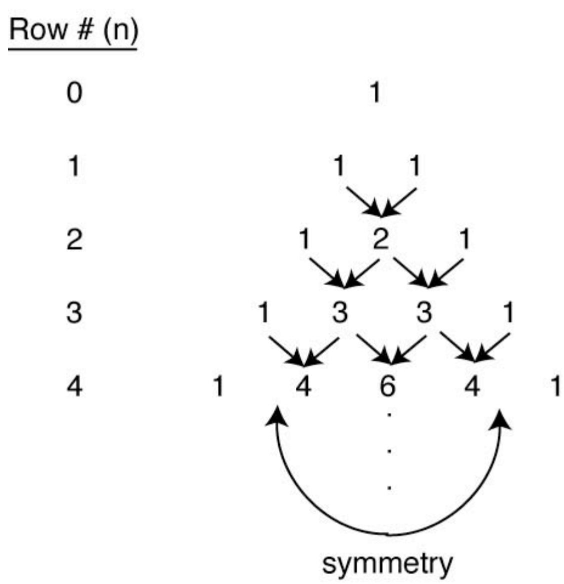
(# terms = power + 1)

<u>What about the coefficients?</u> They are given by …

PART B: PASCAL'S TRIANGLE

The ingredients:

"Tent" of "1"s
Any other entry = sum of the two entries immediately above



Row n gives the coefficients for $(a+b)^n$.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$\vdots$$

(See my website for some magical properties of Pascal's triangle!)

PART C: EXPANDING POWERS OF GENERAL BINOMIALS

Example

Expand and simplify $(3x-y)^3$ using the Binomial Theorem.

Solution

We will use the template for $\left[a+b\right]^3$.

$$
\begin{aligned}
&(3x-y)^3 \\
&=\left[\underbrace{3x}_{\substack{\text{Sub}\\ a=3x}}+\underbrace{(-y)}_{\substack{\text{Sub}\\ b=-y}}\right]^3 \\
&=[a+b]^3 \\
&=a^3+3a^2b+3ab^2+b^3 \\
&\qquad \text{Sub back: } a\leftarrow(3x),\ b\leftarrow(-y) \\
&=(3x)^3+3(3x)^2(-y)+3(3x)(-y)^2+(-y)^3 \\
&\qquad \text{Simplify. First, do powers.} \\
&=27x^3+3\left(9x^2\right)(-y)+3(3x)\left(y^2\right)-y^3 \\
&=27x^3-27x^2y+9xy^2-y^3
\end{aligned}
$$