MATH 141: PRECALCULUS

A COMPANION TO PRECALCULUS BY LARSON / HOSTETLER (7TH EDITION)

KEN KUNIYUKI
SAN DIEGO MESA COLLEGE

CONTACT INFORMATION

Ken Kuniyuki:

Email address: kkuniyuk@yahoo.com

Website: http://www.kkuniyuk.com (various course notes and exams are available)

Feel free to send emails with suggestions, improvements, tricks, etc.! I welcome them!

LICENSING

PARTIAL BIBLIOGRAPHY / SOURCES

Algebra: Blitzer, Lial, Tussy and Gustafson

Precalculus: Larson and Stewart

Calculus: Larson, Stewart, and Swokowski

Complex Variables: Churchill and Brown, Schaum's Outlines

Discrete Math: Rosen

Harper Collins Dictionary of Mathematics

Wikipedia

People: Larry Foster, Laleh Howard, Terrie Teegarden, Tom Teegarden (especially for the Frame Method for graphing trig functions), and many more.

TABLE OF CONTENTS

COLOR CODING

Notes in red: Warnings!

Notes in green: Technical and historical notes, applications in Calculus, etc.

Notes in blue: References to the textbook (which may change with different editions), other parts of this work, my web site, etc.; and information specific to San Diego Mesa College and / or my class.

Colors may be used in figures without those interpretations.

TECHNOLOGY USED

This work was produced on an eMac with Microsoft Word, MathType, Adobe Illustrator, Adobe Acrobat, and Mathematica and Calculus WIZ.

PRELIMINARY TOPIC: LOGIC

"Logic is the science of the necessary laws of thought." – Immanuel Kant, philosopher
"Logic is the anatomy of thought." – John Locke, philosopher.

Why bother? Although logic is a subject that is often relegated to Discrete Math courses (such as Math 245 at Mesa) and courses in computer science and electrical engineering, its fundamentals are essential for clear and precise mathematical thought.

PART A: PROPOSITIONS AND "IF-THEN" STATEMENTS

While thinking about your Precalculus class, you consider the following (true!) statement:

"**If** I get an A, **then** I pass."

This statement is of the form "**If** p, **then** q," where:

p is the proposition "I get an A," and
q is the proposition "I pass."

A proposition is a statement that is either true or false.

p is called the hypothesis; it is an assumption or a condition.
q is called the conclusion.

The statement "If p, then q" can be written as "$p \rightarrow q$."
However, the "$\rightarrow$" arrow tells us nothing about whether the statement is true or false.
Because we know the statement to be true, we can write "$p \Rightarrow q$."
The "$\Rightarrow$" may be read as "implies."

Unless we suspect someone is incompetent or trying to trick us (like on a True-False exam), we generally assume that if-then statements given to us in mathematics books and in lectures are true.

Warning: The "$\rightarrow$" arrow is also used as a notation for "limit" in Calculus.
The use of "$\Rightarrow$" helps us avoid the overuse of "$\rightarrow$."

PART B: WHEN IS AN "IF-THEN" STATEMENT TRUE?

Consider the statement:

"**If** a number is an integer, **then** that number is rational."

This is a true statement, because **every** integer is also a rational number.

For example, the integer 7 can be written as $\frac{7}{1}$. Because 7 can be expressed in the form $\frac{\text{integer}}{\text{nonzero integer}}$, it must be a rational number.

This observation can be extended (or generalized) to **all** integers.

Now, consider the converse of the above statement:

"**If** a number is rational, **then** that number is an integer."

This is a false statement, because we can find a counterexample.
A counterexample describes a situation in which the hypothesis is true, but the conclusion is false.

Here, we may observe that $\frac{1}{2}$ is a rational number, yet it is not an integer.

Warning: The discovery of even one counterexample can be used to disprove a statement (i.e., to prove that a statement is false). However, a single example is usually not enough to prove that a statement is true! To prove that a statement is true, one often needs to present a rigorous and general argument that applies to all cases where the hypotheses hold.

Of course, there exist rational numbers such as 7 and -3 that are also integers.

Because the converse is false, we can write the following:

(A number is rational) $\not\Rightarrow$ (That number is an integer),

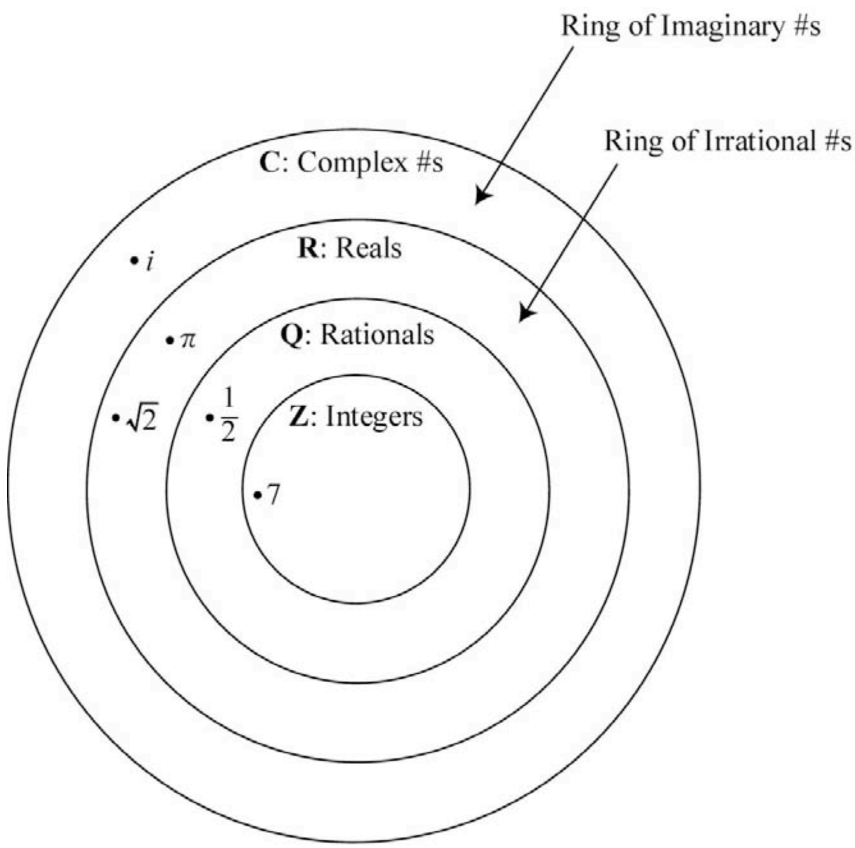
where "$\not\Rightarrow$" denotes "does not imply."

PART C: VENN DIAGRAMS AND SUBSETS

Historical Note: These were named after the English logician John Venn (1834-1923).

Venn diagrams, which are often used in set theory, can help us understand if-then statements such as the ones in Part B.

Here is a Venn diagram for sets of numbers:



How does this diagram confirm what we said about our if-then statements in Part B?

Technical Note: Because every integer is also a rational number, we say that the set of integers (denoted by "**Z**") is a **subset** of the set of rational numbers (denoted by "**Q**"). We write: $\mathbf{Z} \subseteq \mathbf{Q}$. Interpret the diagram correctly: **Z** is contained within **Q**, and so on.

Note: **Z** comes from "Zahlen," the German word for "integer." **Q** comes from "Quotient."

We will let $\mathbf{Z}_+$ denote the set of positive integers, $\mathbf{Z}_-$ denote the set of negative integers, $\mathbf{Z}_{\geq 0}$ denote the set of nonnegative integers (which consists of the positive integers and 0), and so forth for **Q** and **R** (but not **C**).

PART D: SET NOTATION

$\forall$ (the universal quantifier) denotes "for all" or "for every."

$\exists$ (the existential quantifier) denotes "there exists" or "there is."

$\in$ denotes "is a member of" or "that is a member of." Think: "is in."

Examples

	Clear English translation
$(x \in \mathbf{Z}) \Rightarrow (x \in \mathbf{Q})$	Every integer is a rational number.
$\forall x \in \mathbf{R},\ x < x + 1$	Every real number is less than one added to itself.
$\forall x \in \mathbf{R},\ \exists y \in \mathbf{R}$ such that $x = 2y$	Every real number is twice some real number.

Note: "$\forall x$" is often interpreted as "$\forall x \in \mathbf{R}$," unless otherwise specified.

Calculus Note: Rigorous definitions for limits may be written using these symbols.

PART E: "IF AND ONLY IF" (OR "IFF") STATEMENTS

If p implies q, and if q implies p, then p is true if and only if (or iff) q is true.
Using arrow notation: If $p \Rightarrow q$, and if $q \Rightarrow p$, then $p \Leftrightarrow q$.

When this happens, we refer to the propositions p and q as equivalent.
Either both are true, or both are false.

Example

$$p: \quad 2x = 6$$
$$q: \quad x = 3$$

$$p \Leftrightarrow q$$

Technical Note: A proposition is a statement that is either true or false. We will call an equation such as "$2x = 6$" a proposition, even though its truth depends on the value of x. Some books differ.

Definitions essentially work as "if and only if" statements.

Example

A number is a rational number **if and only if** it can be written in the form

$$\frac{\text{integer}}{\text{nonzero integer}}.$$

True "if and only if" statements arise when an "if-then" statement **and its converse** are both known to be true.

PART F: CONVERSE, INVERSE, AND CONTRAPOSITIVE

The converse of $p \to q$ is: $q \to p$

The inverse of $p \to q$ is: $\sim p \to \sim q$ (where "~" denotes "not")

The contrapositive of $p \to q$ is: $\sim q \to \sim p$

Technical Note: "$\neg$" is also used to denote "not." It is a **negation** operator.

Think: Take the original statement, and "switch and negate."

> Contrapositive Theorem:
>
> If an "if-then" statement is true, then its contrapositive **must** be true, and vice-versa. In other words, they are logically equivalent.

Another way of putting it:

$$(p \to q) \Leftrightarrow (\sim q \to \sim p).$$

This can be proven using truth tables in a Discrete Math class (Math 245 at Mesa).

As a result, any "if-then" theorem you find in a (good) book has a contrapositive theorem associated with it that you get for free.

Warning: An "if-then" statement is neither logically equivalent to its converse nor to its inverse. (However, the converse and the inverse **are** logically equivalent to each other! Why? How are they related?)

Example 1

Recall the if-then statement:

$$\textbf{If } \underbrace{\text{I get an A}}_{p}, \textbf{ then } \underbrace{\text{I pass}}_{q}.$$

Of course, this statement is true.

What is the converse? Is this statement true or false?
What is the inverse? Is this statement true or false?
What is the contrapositive? Is this statement true or false?

Solution

Original if-then statement:

$$\textbf{If } \underbrace{\text{I get an A}}_{p}, \textbf{ then } \underbrace{\text{I pass}}_{q}.$$

This statement is **true**.

Converse:

$$\textbf{If } \underbrace{\text{I pass}}_{q}, \textbf{ then } \underbrace{\text{I get an A}}_{p}.$$

This statement is **false**, because we can find a counterexample.
I can pass the class with a grade of B or C.

Inverse:

$$\textbf{If } \underbrace{\text{I do not get an A}}_{\sim p}, \textbf{ then } \underbrace{\text{I do not pass}}_{\sim q}.$$

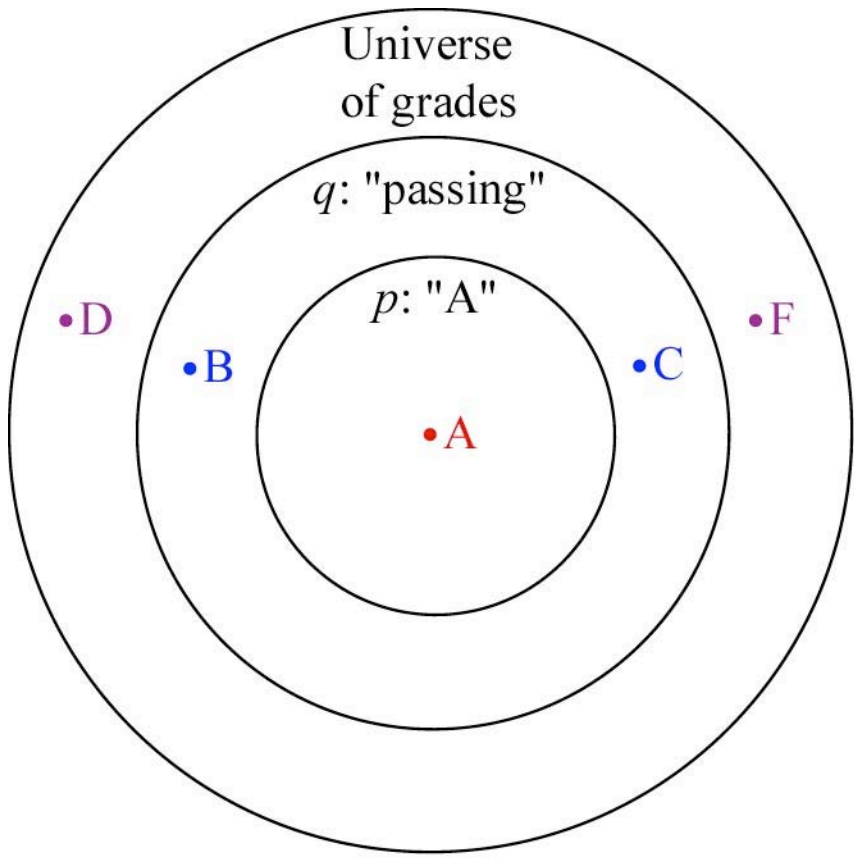
This statement is **false**, because we can find a counterexample.
I can have a grade of B or C, and I can pass.

Contrapositive:

If $\underbrace{\text{I do not pass}}_{\sim q}$, **then** $\underbrace{\text{I do not get an A}}_{\sim p}$.

This one's **true**. By the Contrapositive Theorem, it **must** be true if the original statement is.

A Venn diagram may clarify matters:
(We assume there are no modifiers such as '+' or '−.')



Example 2

Consider the statement:

$$\textbf{If } \underbrace{\text{I get an A, B, or C}}_{p}\text{, }\textbf{then } \underbrace{\text{I pass}}_{q}.$$

Its converse is:

$$\textbf{If } \underbrace{\text{I pass}}_{q}\text{, }\textbf{then } \underbrace{\text{I get an A, B, or C}}_{p}.$$

This time, the converse is **true**.

Therefore, the following "if and only if" statement is true:

$$\underbrace{\text{I get an A, B, or C}}_{p}\ \textbf{if and only if}\ \underbrace{\text{I pass}}_{q}.$$

PART G: NECESSARY AND / OR SUFFICIENT CONDITIONS

Example 3 (going back to Example 1)

Recall the if-then statement:

$$\textbf{If } \underbrace{\text{I get an A}}_{p}\textbf{, then } \underbrace{\text{I pass}}_{q}.$$

Is p sufficient for q?

Yes, because $p \Rightarrow q$.

Think: If I get an A, then that would be enough for me to pass.

Is p necessary for q?

No, because $q \not\Rightarrow p$.

(The answer would have been "Yes" if $q \Rightarrow p$.)

Think: I don't need to get an A to pass.

Example 4 (going back to Example 2)

Recall the "if and only if" statement:

$$\underbrace{\text{I get an A, B, or C}}_{p} \textbf{ if and only if } \underbrace{\text{I pass}}_{q}.$$

This is true, so $p \Leftrightarrow q$, and we say that p is necessary and sufficient for q. Therefore, p and q are equivalent statements.

PART H: REVERSIBILITY

Example 5

You have surely written the following in an algebra class:

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$\text{(By the "Square Root Method":)}$$
$$x = \pm 2$$

All three of these statements are equivalent. We could have written:

$$\begin{aligned} & x^2 - 4 = 0 \\ \Leftrightarrow \quad & x^2 = 4 \\ \Leftrightarrow \quad & x = \pm 2 \end{aligned}$$

The sequence of steps here was completely reversible. You could have started at the bottom statement and worked your way up through equivalent statements up to the top.

Example 6

Now consider the following sequence of statements:

$$x = 2$$
$$x^2 = 4$$
$$x^2 - 4 = 0$$

Be aware that $x = 2$ and $x^2 = 4$ are **not** equivalent statements, because the step of squaring both sides of an equation is **not** reversible here. Observe that $x^2 = 4$ does **not** imply that $x = 2$, because $x = -2$ is also possible. We cannot use "⇔" for this first step, though we can still use "⇒."

$$\begin{aligned} & x = 2 \\ \Rightarrow \quad & x^2 = 4 \\ \Leftrightarrow \quad & x^2 - 4 = 0 \end{aligned}$$

Warning: In your Algebra II class (Math 96 at Mesa), when you were solving radical equations with even roots (such as square roots), you were supposed to check your tentative solutions in the original equation to make sure that they panned out. This was because the step of squaring both sides of an equation is often not reversible.

The net result of this sequence of statements is that, if $x = 2$, then $x^2 - 4 = 0$. Since we are only reading the statements in one direction (i.e., down) in that case, we may want to replace the "⇔" with "⇒" in the last step:

$$\begin{aligned} & x = 2 \\ \Rightarrow \quad & x^2 = 4 \\ \Rightarrow \quad & x^2 - 4 = 0 \end{aligned}$$

There is often no point in using "⇔" as opposed to "⇒" unless you have complete reversibility through the list of statements, or if you want to carefully analyze your argument.

If you do not use "⇒" or "⇔" in your work, you should at least be able to recognize which of your steps are reversible, and which are not.