Simpson Symmetrized and Surpassed

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Simpson's rule is a well-known numerical method for approximating definite integrals. It is named after Thomas Simpson, who published it in 1743, although it was known already more than a century before that. Bonaventura Cavalieri gave a geometric version of Simpson's rule in 1639, and James Gregory published the rule in 1668. Others who published not only Simpson's rule but also more general formulas before Simpson's publication in 1743 include Isaac Newton, Roger Cotes, and James Stirling [**4**, p. 77].

Many calculus textbooks state Simpson's rule in its composite form, which says that

$$\int_a^b f(x)\,dx \approx \Delta x\left[\frac{1}{3}f(x_0)+\frac{4}{3}f(x_1)+\frac{2}{3}f(x_2)+\frac{4}{3}f(x_3)+\cdots\right.$$
$$\left.+\frac{2}{3}f(x_{n-2})+\frac{4}{3}f(x_{n-1})+\frac{1}{3}f(x_n)\right],$$

where n is a positive even integer, $\Delta x = (b-a)/n$, and $x_i = a + i\Delta x$ for $0 \le i \le n$.

Simpson's rule is surprisingly accurate. For example, although it is based on approximating the function f on various intervals with quadratic polynomials, it is exactly correct even if f is a cubic polynomial. However, the asymmetric treatment of the even- and odd-numbered sample points that results from the alternation of the coefficients $4/3$ and $2/3$ seems counterintuitive. The sample points are evenly spaced between a and b, so once one gets away from the endpoints of the interval, every sample point looks very much like every other one. Why should adjacent sample points be treated so differently? Others have raised the same issue before. For example, Roger Pinkham [**8**, p. 92] argues that "... the function evaluations in the middle of the interval are on an equal footing. One feels that they should be treated evenhandedly."

In this paper I will show that symmetrizing the treatment of even- and odd-numbered sample points in Simpson's rule can lead to more accurate approximate integration formulas. These formulas will still be based on approximating f on intervals with quadratic polynomials, and they will still be exact for cubic polynomials. However, the error bounds will be smaller than the error bound for Simpson's rule, and all coefficients except for a few at the beginning and end will be equal to 1.

It is not my intention in this paper to study numerical integration in general. Rather, I will focus on the limited topic of Simpson-like numerical integration rules that are based on quadratic approximation. My question is not whether Simpson's rule is the best way to approximate definite integrals, but rather whether Simpson's rule, with its asymmetric treatment of even- and odd-numbered sample points, is the best way to employ quadratic approximations in numerical integration.

A first attempt

It will be helpful to begin by reviewing briefly the derivation of Simpson's rule for approximating $\int_a^b f(x)\,dx$. The first step of this derivation is to divide the interval